Comparing two algebraic approaches to calculus: WIC Prelude and COTP

Comparing the approaches by (1) Michael Range in AMM 2011, Notices 2014 and "What is Calculus?" 2016 (only "Front Matter" and "Prelude"), and (2) Colignatus in "A Logic of Exceptions" 2007 and "Conquest of the Plane" 2011

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Summary

Michael Range (2016) "What is Calculus?" (**WIC**) and its chapter "Prelude" and Thomas Colignatus (2011) "Conquest of the Plane" (**COTP**) are *proofs of concept*, that show how one might implement a course in calculus starting with an algebraic approach and avoiding limits as long as possible. A *proof of concept* comes with *notes for instructors* and *discussion of didactics*, but not all is explained, since the idea is to *show* how it works. Thus, evaluations by others are useful to highlight not only the explicit explanations but also the actual (implicit) implementations that only transpire from following the method step by step. In the present discussion, teachers and other readers will find information about WIC Prelude that cannot be found in the WIC "Notes for instructors".

This comparison concerns the algebraic approach to calculus and thus concerns the WIC Prelude and not the main body of WIC. WIC shows an approach without awareness or reference to COTP. As author of COTP I may have a bias but I will try to evaluate WIC Prelude as unbiasedly as possible. This discussion should highlight aspects of COTP as well. The reader is invited not to mistake this highlighting as sign of bias.

WIC claims this readership: "Undergraduates, high school students, instructors and teachers, and scientifically literate readers with special interest in calculus and analysis." This would be too ambitious. The WIC Prelude relies on (group theory) notions of "rational function" and "polynomial theory" that will only fit the matricola of science students and up. On the other hand, COTP is a primer and thus targets teachers and researchers of didactics. It only relies on notions for non-mathematics majors in matricola and highschool and thus can support such students as well.

(1) Since the 17th century mathematics has focused on the dual nature of slope (derivative) and area (integral). This discussion is entirely missing in the Prelude (though not in the body of WIC). In COTP, there is the joint development of both integral and derivative, starting with surface and deriving the derivative in reversable steps.
(2) The *tangent* from trigonometry is here the relevant measure of *slope*. Thus there are a *slope of a line* and a *slope of a function*. The hallmark of calculus is that it provides a method to find the slope of a function even at points where it is curved. The notion of the *incline* or *tangent line* is that it adopts as its own slope this slope of the function that has been found by the derivative. A minor didactic problem with the word "tangent" is that its original Latin meaning is "touching", which fits the origins in antiquity when mathematicians started looking at these issues from the notion of touching (like line and circle). COTP uses the standard term of tangent line, though Colignatus (2016g) suggests to *rename* it. The new suggested term is *incline* since an incline (tangent line) can also cut a function and not only "touch" it, see e.g. x^3 at x = 0. WIC Prelude however puts emphasis on this "touching", with reference to antiquity before the new insighs in the 17th century. WIC Prelude doesn't rename but *redefines* "tangent" as a line that causes a double root at the intersection of this very line and the function. This latter definition better be called the *double root line.* Very curiously, WIC Prelude doesn't discuss let alone prove that this double root line actually also gives the slope of the function. Thus the very hallmark of calculus is missing from WIC Prelude. Students learn to find double root lines, that apparently "touch" curves, but this provides only the slope of the line and there is no

discussion or proof that this provides the slope of the function. A major clash for WIC Prelude thus is between the trigonometric tangent and the tangent as "touching".

(3) WIC Prelude claims to avoid limits for algebraic functions. Yet, the Prelude still contains references to limits. This has the effect of a boomerang and creates doubt about the relevance for the algebraic approach. WIC Prelude is unclear about the status of this presentation: (i) Is it only didactics (so that a convincing analysis also for algebraic functions still requires limits) or (ii) is it analytically possible to *really avoid* limits for algebraic functions ? For the latter, WIC Prelude refers to the theory of "rational functions" (and extends to algebraic functions), see Colignatus (2017d). (As said, this reference to rational functions restricts its readership.) Yet the Prelude contains also introductory comments on approximation, continuity and limits, and the very title is "Prelude". The subtitle of WIC is *"From Simple Algebra to Deep Analysis".* This suggests that the author still requires these notions for a *convincing* analysis even on algebraic functions anyhow. An alternative exposition would be to have "Prelude One" that avoids limits as sufficient for algebraic functions (also for the integral), and to have "Prelude Two" for introductory comments on approximation, continuity and limits. On the other hand, COTP instead claims a refoundation of calculus by returning to the notion of algebraic expressions that contain information about functions. Working with expressions and manipulating the domain allows the definition of a "dynamic quotient". This can be used directly for the derivative (in its relation to the integral).

(4) There is the convention to define the line $y = c + s\ x$ so that the slope holds at any x "by definition". This definition puts the notion "the slope at $x = 0$ is s too" into our mind. However, when we look closely at $(y - c) / x = s$, it is fair to wonder about $x = 0$ and in particular about $\{0, c\}$. We cannot calculate it but refer to the definition again. The *definition* now is used to cover up a troublesome issue. This becomes a conceptual trap. However, referring to the definition also involves the recognition that the tools in the toolbox of algebraic operations do not allow us to recover the answer by means of any operation in the present toolbox. It is better to define an operation, that explicitly deals with the troublesome issue. Once we have this operation, we can also use it in the derivative. Let us use the dynamic quotient $(y - c) // x = s$, and see the body of this paper for the clarification that this involves a manipulation of the domain. We are currently dealing with conceptual resistance from the pre-1850 period when the set theoretic notions of variable and domain were not so developed yet.

(5) WIC Prelude actually uses the steps of the dynamic quotient but without developing it formally, see **Appendix E**.

(6) The reference in WIC Prelude to the theory of the "rational functions" (RF) is problematic. This theory has been designed for the purpose of theorems in algebra, notably that the RF (defined in a particular manner) form a field. This theory apparently has not been designed to generate an alternative to Analysis. There are tricky aspects in proofs, notably where a rational function excludes singularities by definition when its denominator would be zero, but such zero value would again be allowed in multiplicative polynomial form. Thus, the reference in the Prelude to the RF is problematic, and the discussion of this is in Colignatus (2017d).

(7) WIC derives the rules of calculus first in WIC Prelude and again in the main body of WIC. This repetition better be avoided.

(8) WIC Prelude claims that there are no algebraic approaches for exponential functions and trigonometry. COTP shows that there are.

(9) WIC and COTP are only *proofs of concept* and thus only provide hypotheses. Didactics are an empirical issue. Students must show what works for them. Advised are randomised controlled trials.

(10) Conventionally the equality sign has symmetry with $a = b$ iff $b = a$. For arithmetic and algebra we accept that $3\ 0$ renders 0. For arithmetic $3\ 0 = 0$ and $0 = 3\ 0$ are fine. Yet in algebra a hidden asymmetry shows up when we look at division. Only nonzero factors are also divisors. Hence $0 = 3\ 0$ makes us aware that the proper statement is (i) For all x: $0 = x\ 0$ or (ii) $0 = \{\text{any } x\}\ 0$. Filling in a particular value $0 = 3\ 0$ is only part of the algebraic solution of $0 = x\ 0$. The dynamic quotient gives symmetry in equality (=) for variables with domains that can be manipulated. See **Appendix H**.

I thank professor R.M. (Michael) Range for the explanations given to my questions. My references here are to the published and/or online available texts only.

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1. Introduction

1.1. Focus

The following comparison will look at these angles on calculus:

- didactics
- content, with the possibility of a fundamental redesign or refoundation of calculus.

We will look at Range (2011) (2014) (2016**bc**, "*Front Matter*" – including "*Preface*" and "*Notes for instructors*" – and "*Prelude*") and Colignatus (2007, "*A Logic of Exceptions*", ALOE, p240-242) (three pages only) and (2011, "*Conquest of the Plane*", **COTP**).

A disclaimer is that I did not evaluate the full book, Range (2016**a**, "*What is calculus?*", **WIC**):

(a) We are interested in the algebraic approach. Hence we have a sound reason to restrict attention to the handling of this approach, which is in the **Prelude** of WIC.
(b) The current focus on the WIC Prelude might do the book injustice, since its author clearly intends it as a Prelude only. The subtitle is: *"From Simple Algebra to Deep Analysis"*. However, in that view, WIC would still be a normal textbook that relies on limits. This is not our interest.
(c) WIC is not open access. I find it important that all should be able to follow this discussion. Thus let me chain myself to the boat and not submit to the siren song that calls for discussing the full WIC and use information that others would not be able to check easily.

Shen & Lin (2014) are critical about current education on calculus too, and they provide a two-hour course (much shorter than a book) in which they also use some algebraic techniques. Colignatus (2017c) discusses this. The effort by Schremmer & Schremmer (1990) to revive Lagrange's algebraic approach must also be mentioned.

The following enlarges on this focus. An overview table is in **Appendix J**.

1.2. Derivative rather than calculus

COTP and Shen & Lin 2014) have the simultaneous development of both integral and derivative, see Colignatus (2017c). In WIC Prelude there is no discussion of the integral, as this is done in the main body of WIC. Thus WIC Prelude is traditional in the separate treatment, and we will look here at only the derivative.

1.3. A stepping stone with risk

WIC Prelude can be defended as an essential approach, and may be an important stepping stone for many mathematicians to even consider an algebraic approach to calculus. The WIC Front Matter and Prelude also shows that Range enjoyed writing it tremendously, and readers will find this engaging. See for example the review van Ruane (2016).

Professor Range's book is challenged by the notion of a "dynamic quotient" in COTP. **Appendix E** shows that the WIC Prelude actually uses the dynamic quotient but without the formal development. The major issue for the algebraic approach thus becomes to clarify what are the steps in the reasoning of the algebraic approach.

Downsides of WIC Prelude are:

- Range puts an unnecessary cap on the algebraic approach. For exponential functions and trigonometry he switches to the standard approach with limits as if there would be no alternative (as in COTP). Thus the algebraic approach is reduced to the phase of intuition and *Prelude* only. The direct use of polynomials and factoring becomes tedious rather

quickly (as also Shen & Lin (2014) mention). Thus, the stepping stone also leads towards a sidetrack and dead-end street. Teachers and didacticians interested in the algebraic approach could be turned down and lose interest.

- The rules of calculus developed in WIC Prelude apparently must be repeated later again for non-algebraic functions. This is unappealing for education.
- Range rekindles Euclid's notion of tangency as "touching" while calculus since the 17^{th} century has moved to the Fundamental Theorem of Calculus (FTC) that slope (derivative) and area (integral) are related inversely. The slope uses the tangent from trigonometry, for which there is no direct connection to "touching". Perhaps there is a necessary connection between algebra and this notion of "touching", but it is confusing to refer to this issue when the major topic is calculus (and not algebra). This focus on algebra with a double root may also be a stumbling block for a simultaneous treatment of both integral and derivative (as in COTP and Shen & Lin (2014)).
- WIC Prelude mentions the difference quotient quite late on p13 in formula P.2 for average velocity (here **Appendix E**), and then on p40 for his fig. 9 (here Figure 9). The Prelude makes a conceptual distinction between difference quotient and limits (numbers) and the double root (algebra). My idea is that students are better served with an understanding of the tangent by means of the quotient – which is also algebra. The quotient is the fundamental notion and not the double root, though Euclid with line and circle is the exception. **Thus, a major issue w.r.t. WIC Prelude is the clash between tangent from trigonometry and tangent as "touching".**
- WIC Prelude relies on notions of higher algebra and group theory, and in particular the theory of *rational functions*. See Colignatus (2017d), that originally started as a section in this present discussion, but that better was developed and now is presented as a separate paper. See there also for some aspects about the WIC Prelude. It appears that this theory of the rational functions is both too complex for highschool students and imbalanced anyway, see a summary below.
- WIC Prelude contains a discussion of approximation, continuity and limits, as an *introduction* for such treatment in the main body of the book. In terms of didactics, one can understand such a ploy towards preparation, as plowing and seeding before harvesting. There are downsides. First this means a mingling of Algebra with issues of Analysis, so that the student may get confused about which is which. Secondly, there is also a boomerang effect, in that this creates doubt whether the value of the algebraic approach is also merely didactics, and not on content. A key example is in **Appendix I**, where the simple quadratic equation for gravity is used to also introduce notions of approximation and continuity via limits, but also as an "argument" to show that the algebraic solution is correct. Trying to evaluate this, I myself started to doubt whether Range considers algebra *sufficient* for the derivative of algebraic functions or whether he still requires limits for a fully *convincing* deduction. See more on this below. Teachers using this book obviously must already develop their own view, but now with a blurred roadmap. Thirdly, Range misses the opportunity to explain that the use of *expressions* allows the use of "formal continuity" rather than numerical continuity as used in analysis. Overall: it would have been better to have a "Prelude One" on algebra only and then, if desired, a "Prelude Two" with an introduction into approximation, continuity and limits.

This discussion will focus on the issue of "tangency" and "continuity". For the algebraic approach to exponential functions and trigonometry one is referred to COTP and Colignatus (2011b) (2017b). For rational functions one is referred to Colignatus (2017d).

1.4. Proof of concept. Readership. Didactics as empirical science

Range WIC and Colignatus COTP are both "proofs of concept": (i) *proposals for different didactics* and (ii) *direct implementation of those proposals.* Both authors provide meta-comments for readers to understand the proposals. Yet a proof of concept may contain subtle points that take too much time to discuss at a meta-level, for it is better to just *show* it. Thus we will also look at the implementation step by step.

The WIC website claims this readership: "Undergraduates, high school students, instructors and teachers, and scientifically literate readers with special interest in calculus and analysis."

[1] This appears to be too ambitious, see below. My advice would be matricola for science students and up. I would belong to the claimed readership of WIC, yet I find that essential information for teachers and researchers on didactics can now only be found in this present evaluation. COTP is a primer (for training of teachers) and can be used for matricola of non-math majors. It doesn't have "exercises" but if those were supplied then it could be used in highschool as supplementary material.

Didactics is an empirical science, and it are only experiments that will show what methods work for students. This discussion should help to design such (randomised controlled) trials.

1.5. Analytic geometry and calculus

COTP is a primer on both analytic geometry and calculus. Thus where WIC Prelude might hold that it only looks at calculus, COTP embeds it within analytic geometry. This reason of embedding is that there are also proposals for some changes there.

Ancient Greek geometry is called *synthetic* since one generates proofs by "putting together" the various givens (definitions, axioms and earlier theorems). In *analytic geometry* one provides proofs by decomposing (analysing) issues in terms of algebra. A subsequent historical development was that even analytic geometry and its system of co-ordinates was seen as not exact enough, whence one looked for foundations in arithmetic. This became the field of *analysis.* Corner stones of the latter are notions of numerical continuity and limits. The current perception in mathematics is that calculus can only be done in *analysis*.

Are we allowed to refer to insights from analytic geometry (or "geometric intuition") as part of proofs ? Mathematicians might grant that this might be done in didactics. Didacticians might grant that mathematicians have the job to question details, which they do in research mathematics. (Thus school mathematics (SM) versus research mathematics (RM).) Views on this might clash when there is the emphemeral notion as if we would withold students essential information by referring to analytic geometry (and some "geometric intuition") instead of requiring that they should be trained to become research mathematicians themselves too. My approach to this is that a sound training in empirical science, and in what is called "applied mathematics", would be the best basis to judge about how to create balance between what is both didactically effective (empirically) and mathematically required. Research mathematics can speak their mind but should not decide upon math education in highschool and matricola for non-math-majors.

1.6. Limit versus evaluation

Notions of continuity and limit make calculus complicated, both on content and in didactics, see Figure 1 for Range (2016).

My comments on this:

(1) In Holland a bit less that 11% of highschool graduates has the Math B (beta) profile with quite a bit of calculus preparing for university. Forty years ago demands were tougher and nowadays the limit is mentioned by handwaiving, and the focus is on mastering rules and applications. Bressoud (2004) gives some information about the USA and the link from highschool to tertiary education. Beta students like in Holland or in the USA with Advanced Placement will not have quite such difficulties as Range refers to.

(2) The real problem is indeed, what he refers to, the confusion about the *need* for limits. The confusion lies not with the students but with the mathematicians. There is no need for a limit if it suffices to find the derivative by an evaluation (of a function with manipulated domain).

[1] http://www.worldscientific.com/worldscibooks/10.1142/9448#t=aboutBook

(3) The notion of a limit is not so hard to grasp or explain, see the asymptotes of 1 / *x*. It are rather the current definitions in mathematics that make the limit more complicated than needed. A first step towards deconstruction is in Colignatus (2016b).

(4) An approach of "only teach rules and applications" (even embellished by handwaiving on limits) sacrifices both rigour and understanding what derivative and integral actual are. Thus the stage is set for a major redesign.

Figure 1. Range (2016b:xvi) WIC "Preface"

Unfortunately, the transition from high school mathematics to calculus is not easy. Students are usually exposed to deep new concepts right at the beginning. In particular, important central applications such as variable velocity, slopes of tangents, and more general rates of change and derivatives are introduced by an approximation process that involves "limits" of certain expressions that formally approach the meaningless quotient $0/0$. Therefore it becomes necessary to investigate and understand such "limits" in order to proceed. Algebraic examples involving polynomials, rational functions, roots, and so on, often tend to confuse matters: The limit as the input x approaches the value a, where x must be assumed $\neq a$, is ultimately found—after algebraic manipulations to remove the troublesome zero from the denominator—by what is de facto *evaluation* of an algebraic expression by setting $x = a$. Thus limits tend to get mixed up with evaluation, often leaving one wondering about what seem unnecessary complications. The confusing relationship between limits and evaluation had surfaced already at the origins of calculus in the 17th century, but that did not stop the pioneers from moving forward. The difficulties were only resolved in the 19th century, when mathematicians introduced precise—and necessarily complicated—technical descriptions of *limits*. Since then, these new abstract concepts—in varying degrees of technical detail—have become a major component of any introduction to calculus. Even when discussed in intuitive non-technical language, they present quite a challenge right at the beginning for anyone who wants to learn and understand calculus.

1.7. Refoundation of calculus versus only redesign in didactics

The approaches by Range and Colignatus have in common that they avoid limits and rely on algebra, both with the claim that they do not sacrifice rigour and understanding. See Figure 2 for Range.

Figure 2. Range (2016b:xvi-xvii), "WIC Preface"

In this book we present a more elementary approach to derivatives for *algebraic* functions that completely avoids limits. More advanced concepts are only introduced later, when algebraic methods no longer work, for example while studying exponential functions. The heart of the matter is an up-to-date version of a fundamental idea that goes back to René Descartes (1596—1650), one of the intellectual giants of his time, and that has remained on the sidelines for centuries.

In more detail, we begin with a *Prelude to Calculus*, in which the ancient tangent problem and some of its variations are introduced and solved for polynomials and other algebraic functions—which are built up by *finite* processes—by using only elementary concepts familiar from high school algebra and geometry. In particular, no mysterious quotients $0/0$ appear, and no limits whatsoever are needed at this stage. Basic rules and formulas are established in a direct and most natural way. The reader thus begins to learn about tangents, derivatives, and all the mechanical rules

of calculus in a familiar setting, without getting burdened by investigations of more advanced concepts based on limits and infinite processes. At

There is a key difference between WIC Prelude and COTP though:

- The set-theoretic approach to functions takes *f* = {{*x,y*} | *x* and *y* in their sets}. The *x* and *y* = *f*[*x*] are *elements* and not quite *variables* (symbols that can be assigned different values). For *analysis*, the sets are (real) numbers. This necessitates notions of *numerical* continuity and limits. Range actually adopts this setting for the body of the book (2016a). These methods are called "more advanced" but – *apparently* – are seen as "fundamentally required" for a *convincing* approach. In the WIC Prelude (2016c), the reference to algebra and polynomials is (*basically*) a didactic tool for lowering the (conceptual) barrier for students, and to link up to the history on notions of tangency. I am a bit at a loss how to regard this. It seems that this is didactics only, and the student still needs the fundamental tools of analysis, to properly treat polynomials too. Perhaps I am wrong, but I found no statement that indicates otherwise. Range's reference to the theory of rational functions may well imply an implicit reliance still on analysis, see Colignatus (2017d). Clearly Range doesn't claim a fundamental redesign of calculus. The subtitle of WIC is *"From Simple Algebra to Deep Analysis",* and thus Range wants to engage supporters of Analysis rather than inform them that their work has become superfluous. Teachers obviously must make up their mind whether the algebraic approach would be sufficient for algebraic functions or whether these would also require limits to arrive at a *convincing* deduction. Yet they better be alerted to the notion that the text does not make it easy for them and the students to decide on this.
- Historically, a function was a *proscription* of how to turn an input into an output, see also Cha (1999). This generated some study of notations and algorithms, as we see nowaday in *computer algebra,* with the algebra of *variables* and *expressions.* Colignatus rekindles this approach. Information about the function is contained in its expression. There is a notion of "continuity in form" (COTP 224-225), using formal expressions rather than numbers. This information can be used when particular methods of arithmetic generate problems, notably with arithmetic division at zero. We can define a notion of "dynamic quotient" that manipulates the domain. This dynamic quotient allows an algebraic definition of the derivative. An algebraic approach is also possible for exponential functions and trigonometry. Looking at calculus in algebraic manner again would be a fundamental redesign, after the Cauchy and Weierstrasz turn to numbers. The approach originated from didactics and it would be up to mathematicians to see how far the redesign can be developed further, see Colignatus (2014) (2017d). The algebra of expressions is less developed indeed, but it seems to me that this is a historical oversight

due to the focus on numerics. I don't think that highschool students should be a victim of this historical development within mathematics. For didactics the use of the dynamic quotient is well-defined, and, the derivative is a mere consequence. Obviously, students majoring in mathematics would have to know both methods.

1.8. Root and root factor. Use "zero" for 0 only

A root x of a polynomial $p[x]$ is a solution of $p[x] = 0$. The root factor will have the form $(x – a)$, for example $x + 1 = x – (-1)$. The term "zero" is used conventionally but somewhat confusingly for both the root and the root factor, and thus it is better to use "zero" for 0 only. Factors need not have roots, for example $x^2 + 1 = 0$ only has complex solutions.

For finding roots, "division" can also be seen as "repeated subtraction". Thus from $x^2 – 1$ we first subtract $(x – 1) x = x^2 – x$ and subsequently $x – 1$ whence we have subtracted $x + 1$ times $x – 1$.

Obviously we should not subtract multiples of 0. When $(x^2 – 1) / (x – 1) = x + 1$ for $x \neq 1$, the question is what to do when $x = 1$. See **Appendix H**.

1.9. Range's redefinition of "tangent" (overview)

The *tangent* from trigonometry is generally taken as a measure of *slope.* There are a *slope of a line* and a *slope of a function*. The hallmark of calculus is that it provides a method to find the slope of a function even at points where it is curved. The notion of the *tangent line* is that it *adopts as its own slope* this slope of the function that has been found by the derivative.

A minor didactic problem with the word "tangent" is that its original Latin meaning is "touching". This fits the origins in antiquity when mathematicians started looking at these issues from the notion of touching (like line and circle). Since the 17th century it was found that the tangent line, based upon trigonometry and derivative, can also cut a function and not only "touch" it, see e.g. x^3 at $x = 0$. Mathematics kept on using the term "tangent line" however. This causes the minor didactic overload to explain to each student that the name is somewhat misplaced.

COTP uses the standard definition of tangent line, though Colignatus (2016eg) suggests to *rename* it. The new suggested term is "incline", as the line that uses the inclination of the function.

WIC Prelude however puts emphasis on this "touching", with reference to antiquity before the new insighs in the 17th century. WIC Prelude doesn't rename but *redefines* "tangent" as a line that causes a *double root* at the intersection of this very line and the function. See his definition in Figure 6 and the discussion there. (Use "double root" and not "double roots".)

This latter definition better be called the *double root line.* Very curiously, WIC Prelude doesn't discuss let alone prove that this double root line actually also gives the *slope of the function.*

- Thus the very hallmark of calculus is missing from WIC Prelude.
- Students learn to find double root lines, that apparently "touch" curves, but this provides only the slope of the line and there is no discussion or proof that this provides the slope of the function.

Awkwardly, readers might think that this is part and parcel of the algebraic approach to the derivative. However, it is a choice by Range to present his Prelude in this manner, and I can only warn that this is didactically confusing.

Thus, in steps:

- In the common vocabulary, the incline (tangent) *by definition* adopts the slope of the curve. This slope is found by means of other methods (notably the derivative).

- The line that intersects with the curve and has a double root better be called the "double root line" (my phrase, not in Range's vocabulary).
- There is nothing in the notion of a double root line that directly links to the slope of the curve. We need an additional theorem to provide this link (**Appendix B or C**). To redefine the term "tangent" to become the double root line does not remove this need. For this proof, there is only the avoidance of limits if one accepts algebraic methods to isolate factors of polynomials. Notions of factorisation and the polynomial remainder theorem (in "pre-calculus") become prerequisite for this approach to teaching calculus.
- The notion of a double root is not necessarily linked to polynomials. The linkage is easier to show for polynomials but it will need investigation for other cases.
- In Range's redefined "tangent" there is no reference to the slope of the curve, whence:
  (1) The notion of derivative (slope) in relation to the integral (area) of the curve disappears. See Colignatus (2017c) for the Shen & Lin (2014) paper that emphasises the link between *derivative as slope* and *integral as height increment*.
  (2) This link can be restored but then requires a theorem, see **Appendix B or C**.
- The common phrase "find the tangent (incline)" asks for the *slope of the curve*, and doesn't ask for the "double root line". Thus the redefinition of "tangent" creates confusion w.r.t. to the accepted vocabulary. A teacher used to the common vocabulary and asking "find the tangent (incline)" might think that her or she asks for the slope of the curve but the student of Range (2016c) will only generate the line with the double root. It wouldn't be clear whether the student can relate this line to the slope of the curve.
- It is advisable to replace "tangent" in the text of the Prelude by the proper term ("double root line"), so that one can see how confusion can arise.
- Didactics gets more complex. One starts with a line that intersects a curve at least twice at two different points. The line is manipulated till the two points overlap at the point of interest, which generates a double root. Range's definition of "tangent" still calls this "intersection", but he wishes to link up to the notion that there is actually a "touching" when these two points overlap. Subsequently, Range must explain again that this "touching" still is intersecting in the case of x^3 at $x = 0$. Thus, the minor didactic problem of explaining that "tangent" also might mean an intersection is now enlarged by the need to explain that intersection might also mean tangent. (Namely when double, though impossible to see that it is double, while the idea of "touching" is that there is *no* intersection but only overlap of function and line.) (Solutions in the complex plane indicate no overlap but may have a higher multiplicity too.)
- A minor example of a source of confusion is: Prelude page 3 has: "Note that for any other line through P that "cuts" the curve – and hence does not fit our intuitive idea of a tangent – the point of intersection really gets counted once." Thus for x^3 there is only one root at $x = 0$? But page 4 then is confusing when there is such a cut and then there is multiplicity of 3. Thus the nomenclature has become more complex.

About Appendices B and C: **Appendix B** uses the notation from the world of polynomials of taking a fixed and x variable. **Appendix C** uses the standard differences Δx and Δf, thus with taking x fixed and Δx variable. This notation is also used in COTP:154 and 224 for the definition of the derivative. It is straightforward to take $\Delta x = x - a$. To compare the appendices with $x = a + \Delta x$, we must rework "x" into "$x + \Delta x$" and "a" into "x".

1.10. Table with the overview of the differences in approaches

Table 1 gives the overview of the differences in approaches.

Table 1. Overview of the differences in approaches

	Limits	*Algebra*	
		"Algebraic functions" [2]	*Expressions*
Double root		Range (2016c:32) has no relation to slope	Sometimes handy [3]
Slope	Standard Range: remains required fundamentally Colignatus: majors in mathematics only	Range (2016c:5): The slope of the curve isn't in the definition of "tangent"	Colignatus (2011:154 & 224): "dynamic quotient" allows to set $\Delta x = 0$ for the derivative (i.e. setting and not just a limit)
Tangent	Definition by trigonometry	Definition by double root	Definition by trigonometry

Remarkably, I propose to change the *name* of "tangent" into "incline", and Range prosposes to change the *meaning* of "tangent" (its definition).

- Common: The tangent for *f* at *x* gives the slope of the curve at {*x*, *f*[*x*]}. [4]
- Colignatus (2016g): The incline (tangent) for *f* at *x* gives the slope of the curve at {*x*, *f*[*x*]}.
- WIC Prelude: The tangent for *f* at *x* gives the double root line for the curve at {*x*, *f*[*x*]}.

Even for mere gravity and the parabola (Prelude, section 4, p11-15), Range still requires continuity (and the latter defined by limits) to justify the derivative (slope of the curve). Thus the reference to "algebra without limits" might well be only *presentation* (didactics) and not *justification*. Also, the announcement that limits are avoided in the Prelude is inaccurate, since this reference to limits for gravity and the discussion of 2^*x* are in the Prelude indeed. Range presents them as an introduction to the main body of the book, but this deviates from the promise of a Prelude without limits.

1.11. Overview of this discussion

This discussion will look at the angles mentioned in the table of contents.

On content, I agree with the actual deductions and many observations by Range. A main comment is that I would add that the dynamic quotient allows a fundamental redesign of calculus. Range's contribution lies in didactics, yet his approach in didactics does not convince me. Obviously, it are the students who must tell what didactics works for them. Thus I am looking forward to classroom experiments, i.e. not just usage of the methods but randomised controlled trials. This present discussion should be helpful to determine what to check.
This present document contains *deliberate repetition.* The same point can be mentioned in different manners. In the final edit, I decided to do so, since the current emphasis is on clarity

[2] Range (2016c:34) here in **Appendix D**, proves that all algebraic functions are continuous, which proof uses a limit for the notion of continuity. The proof in **Appendix B** shows that this continuity is essential for Range's results. Thus the claim that his method doesn't use limits is dubious. The double root line can be found without limits but the trigonometric tangent required for the slope of the function still relies on it. See also Colignatus (2017d) that the theory of rational functions in its current approach tends to rely on analysis with notions of limits and densities to remove singularities.
[3] Since the error function ε[*x*] (**Appendix B**) or ε[*x,* Δ*x*] (**Appendix C**) contains unknown slope *s,* algebraic methods like comparing coefficients may generate solutions for *s* faster than standard deductions, but one still would need a link to the slope of the curve. Thus, a method of calculation should not be confused with a fundamental relationship.
[4] Thus, this meaning comes from trigonometry, that only uses the metaphor of "touching". In antiquity, the notion concerned touching, but this has changed after the 17th century.

and not on speed. The different phrasings should help to see the same points in slightly different lights and/or combinations with other points.

It is useful that the reader knows about the genesis of this document. When I started with Range's work (op. cit.) I had no deep knowledge about Descartes's method with the circle, Ruffini's Rule on polynomial coefficients, and the interplay between the theory of polynomials and the theory of rational functions. It was only because of Colignatus (2016efg) that I discovered Range's work (op. cit.) at the end of December 2016. It took some effort to dissect *both* WIC Prelude's approach in didactics *and* the underlying theory that it relies on. A student would have to take the textbook at face value and do the exercises, but a teacher and student of didactics needs more to understand what is being done (and blissfully skips the exercises). Given that WIC is a *proof of concept* apparently not all relevant information was in the *Notes for instructors.* Thus, it took an effort to recover the main points, now given here. Colignatus (2017d) is a spin-off of this effort. In the various stages in this dissection, the same points got rephrased, also to link up with new angles. This explains the repetition that now shows up in the final edit. As stated, this repetition deliberate, and one hopefully sees the value of this.

The definition of the dynamic quotient can be found in Colignatus (2007:241) or (2011:57). It appears that reference is not sufficient and that some readers require that a discussion is self-contained. This condition is awkward since COTP really deserves a study because of its approach to didactics and essential refoundation of calculus. Yet, a section below thus repeats the definitions of the dynamic quotient and the derivative that uses this.

2. Theory of rational functions. Linking up to school mathematics

The theory of rational functions (RF) starts with this notion:

> "A rational function is a function $w = R(z)$, where $R(z)$ is a rational expression in z, (...). A rational function can be written (non-uniquely) in the form $R(z) = P(z) / Q(z)$ where P, Q are polynomials, $Q(z) \neq 0$." [5]

It appears that there are two versions of this theory:

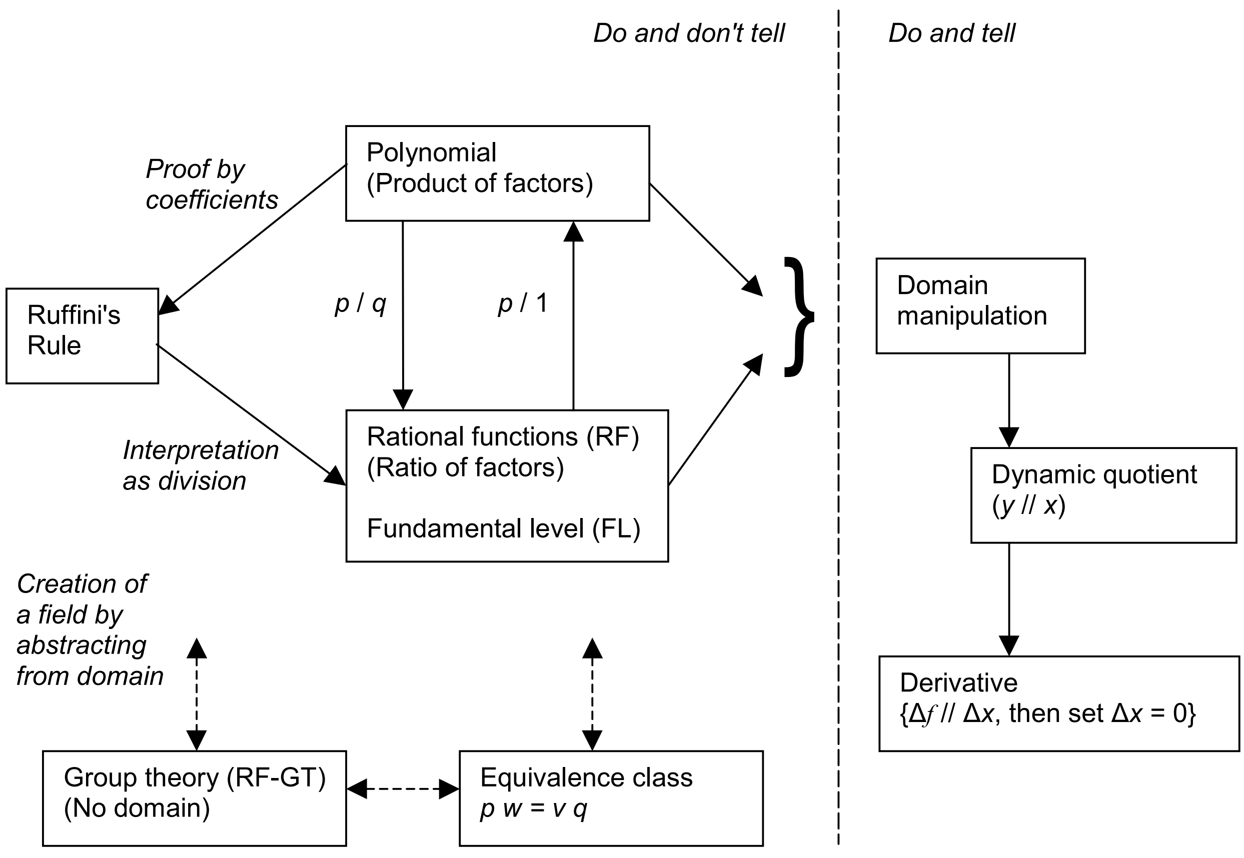
(1) RF-GT: There is the version falling under *group theory*, in which the polynomials have no domain, and in which $Q[z] = 0$ would be the zero-polynomial (with the value 0 for all z). Also, z is called an *indeterminate* rather than *variable,* but let us use *nondetermined.* [6] RF-GT finds that the rational functions form a field. (If these theorists agree that y / x normally applies to domains, then they would have to agree that they need a new symbol for the RF[y, x] rather than abusing the same symbol for other purposes. Paul Garrett suggests to replace "ratio" by "pair". [7])
(2) RF-FL: There is a *fundamental level* version for matricola for non-math majors and perhaps highschool, in a course belonging to "pre-calculus", in which the polynomials have domains, and in which $Q[z] = 0$ would be solved for roots in the domain, in order to exclude those values for the ratio. (See for example Juha Pohjanpelto. [8]) This version of RF causes the notion of "removable singularity" – which notion would not be relevant if there would be no domain. [9]

Colignatus (2017d) discusses the theory of rational functions. Some relevant references to Range (2016c) WIC Prelude have been moved there too. This theory of RF has been designed for the purpose of theorems in algebra, notably that the RF (defined in a particular manner) form a field. This theory apparently has not been designed to generate an alternative to Analysis. There are tricky aspects in proofs, notably where a *rational function* excludes singularities by definition when its denominator would be zero, but such zero value would

[5] https://www.encyclopediaofmath.org//index.php?title=Rational_function&oldid=17805
[6] In Wolfram's *Mathematica* the term *Indeterminate* stands for undefined, comparable to Infinity. Above group theory is better served with "nondetermined".
[7] http://www-users.math.umn.edu/~garrett/m/algebra/notes/06.pdf
[8] http://oregonstate.edu/instruct/mth251/cq/FieldGuide/rational/lesson.html
[9] https://www.encyclopediaofmath.org/index.php/Removable_singular_point

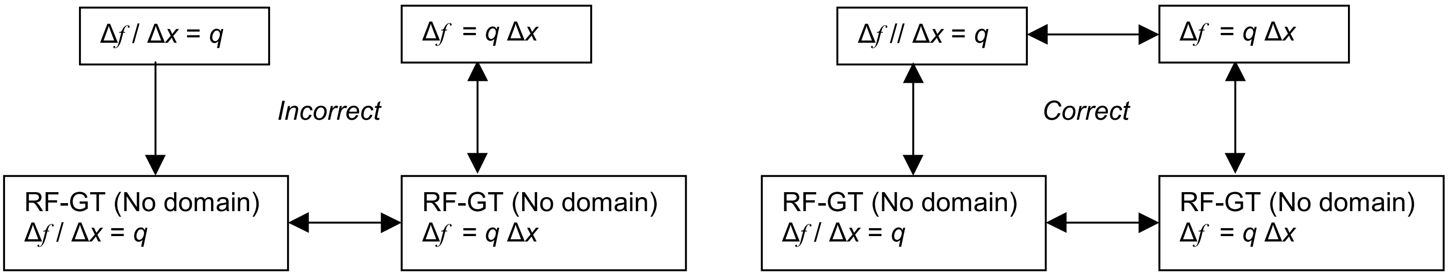
again be allowed in *multiplicative polynomial* form. Thus the domain is manipulated but one doesn't draw attention to this. Not every factor is also a divisor, see **Appendix H**. I was myself unware until last month of the general proof of the polynomial remainder theorem that excludes roots. Working in a Ring without division is a nice trick to achieve some results. But these results must also be translated, for our purposes, to the derivative for the reals, a field. Then the slope enters the discussion, which uses the tangent from trigonometry. It appears that the group theory approach is less relevant. Results still must be translated to functions that have domains, but it appears that authors gloss over this requirement. ((1) They might see this as Analysis, while they themselves are interested in Algebra. (2) In conventional view, it is Analysis indeed, so that conventionally this manipulation of the domain isn't Algebra but still relies on limits. (3) It is COTP that refers to a theory of expressions so that the manipulation can be based upon Algebra only.) RF-GT is irrelevant for highschool anyway. Subsequently, the factoring of polynomials in multiplicative form doesn't quite explain the property of the slope, that has the ratio format, namely from the tangent in trigonometry. Overall, the theory of rational functions appears to be rather evasive on the issue of division by zero. It is better to be explicit about the manipulation of the domain. This is achieved by the dynamic quotient. Figure 3 and Figure 4 give an overview of these relations, taken from Colignatus (2017d).

Figure 3. Overview of relations (taken from Colignatus (2017d))



PM. This reference to rational functions is not to be confused with Range's proper treatment of rational functions, see **Appendix F**.

Figure 4. Steps without a memory where they originated (First line with real domain)



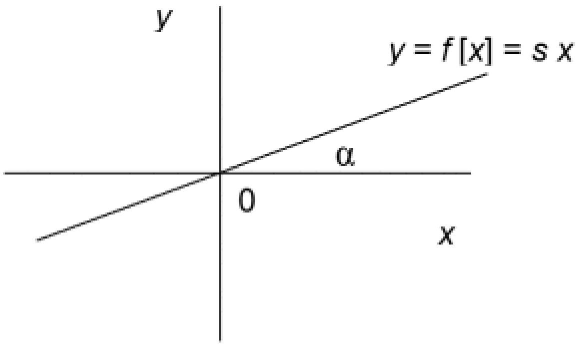
3. Short restatement of dynamic quotient and derivative

The following basically repeats sections from Colignatus (2016ad) (2017d). See COTP for the theoretical development and the approach to calculus in general (integral and derivative).

3.1. Ray through the origin and definition of dynamic quotient

Let us consider a ray – rays are always through the origin – with horizontal axis x and vertical axis y. The ray makes an angle α with the horizontal axis. The ray can be represented by a function as $y = f[x] = s\, x$, with the slope $s = \tan[\alpha]$. Observe that there is no constant term ($c = 0$). See Figure 5.

Figure 5. A ray with angle α and slope *s*



The quotient y / x is defined everywhere, with the outcome s, except at the point $x = 0$, where we get an expression 0 / 0. This is quite curious. We tend to regard y / x as the slope (there is no constant term), and at $x = 0$ the line has that slope too, but we seem unable to say so.

There are at least five responses:

(i) The argument can be that y has been defined as $y = s\, x$, so that we can always refer to this definition if we want to know the slope of the ray. This approach relies on a notion of a "memory of definitions", to be used when algebra lacks richness in expressiveness.

(ii) Standard mathematics can take off with limits and continuity.

(iii) A quick fix might be to redefine the function with a branching point:

$$\text{SlopeOfRay}[y, x] = \begin{cases} y / x = s & \text{if } x \neq 0 \\ s & \text{if } x = 0 \end{cases}$$

We can wonder whether this is all nice and proper, since we can only state the value *s* at 0 when we have solved the value elsewhere (or rely on the definition as in (i) again). If we substitute *y* when it isn't a ray, or example $y = x^2$, then we get a curious construction, and thus the definition isn't quite complete, since there ought to be a test on being a ray. Anyway, defining lines in this manner isn't a neat manner. It is really so, that we cannot define a line as $y = s\ x + c$ and that we must specify the branching when $x = 0$?

(iv) The slope y / x is regarded as a special case of "rational functions". See the section above and the discussion of Range (2016c:16) in Colignatus (2017d). If we work on coefficients only, then we get Ruffini's Rule (a case of "synthetic division"), see Colignatus (2016ef) also referring to MathWorld. [10] The first problem is that in this approach the issues of "identifying the factors" and "adjusting the domain" are only indicated and not made explicit via separate notations. The term "synthetic division" indicates that it might not be "proper division". To what extent is there proper division, so that "eliminating" the factor *x* generates a result that can be understood as the slope of the line at that point (i.e. fitting to the tangent in trigonometry) ? The second problem is that this remains within the realm of polynomials.

(v) The algebraic approach uses the following definition of the dynamic quotient. Let y / x be as it is used currently in textbooks, and let $y // x$ be the following process or program, called *dynamic division* or *dynamic quotient*, with numerator *y* and denominator *x*:

> $y // x \equiv$ { y / x, unless *x* is a variable and then: assume $x \neq 0$, simplify the expression y / x, declare the result valid also for the domain extension $x = 0$ }

Thus in this case we can use $y // x = s\ x // x = s$, and this slope also holds for the value $x = 0$, since this has now been included in the domain too.

We thus *extend the vocabulary of algebra*, so that *multiplication with variables* gets an inverse with *dynamic division by variables*. Since this is a new suggestion we must obviously be careful in its use, but the application to the derivative is a case that appears to work.

The case of the line may be seen as a special case of a polynomial. However, the general notion is "simplify", and there might be other ways than just eliminating factors.

3.2. Dynamic quotient has the denominator as a variable

Simplification only applies when the denominator is a variable but not for numbers. Thus $x // x = 1$ but $4 // 0$ generates $4 / 0$ which is undefined. Also x / x is standardly undefined for $x = 0$.

This definition assumes a different handling of different parts of the domain. The test on the denominator is a syntactic test. When the denominator is an expression like $(p + 2)$ then the syntactic test shows that the denominator is a variable, $x = p + 2$. One does not substitute "$(p + 2)$ is a variable" for substitution doesn't look at syntax but uses the value of the variable.

It has been an option in the {...} definition above to write "(a) variable" instead of "a variable", which allows a shift from the syntactic test towards the semantic test of variability, and which also allows *substitution* into the definition, like "$(p + 2)$ is (a) variable". After ample consideration, already in 2007 and later explicitly in Colignatus (2014), I think that we are better served with the syntactic test on the denominator, since this directly leads to the question: what is the domain of the denominator ?

[10] http://mathworld.wolfram.com/RuffinisRule.html

The use of the curly brackets {...} also borrows from *Mathematica.* The brackets signify a list, that can be a set, but when the elements are expressions then the sequential evaluation of those turns into a programme.

3.3. From eliminating factors in polynomials to general "simplification"

In multiplication, (x – 1) (x + 1) = (x^2 – 1) holds for all real x. For division we lack an efficient vocabulary to express (x – 1) = (x^2 – 1) / (x + 1), since this is undefined for x = -1. We can introduce branching, but still would have to use a limit to recover the value at x = -1. When we want to identify or *isolate* the factors however then this "isolate" would commonly be tantamount to requiring division.

An alternative way to identify factors (and find the derivative) for polynomials is the use of coefficients and Ruffini's Rule. If multiplication for polynomials is equivalent to manipulating coefficients, then the latter can also be used for the reverse process of division. See Colignatus (2016ef), that was inspired by (with thanks to) Harremoës (2016) also linking to Bennedsen (2004). It works for polynomials but is it general enough, for non-polynomials ?

There remains the notion of a slope however too. There is no clear link between coefficients (Ruffini's Rule) and the slope. We find the proper *values*, which suggests that there is such a link, yet this link must be shown. The method may be an efficient calculation method, but it doesn't explain that when we find outcome *s,* then we may also declare that it is valid for x = 0 (for we cannot do 0 / 0). Ruffini's Rule suggests that the user sets up a division, y / x, but when we look at the proof why it works, [11] then we see addition and multiplication, and thus division (or repeated subtraction) is only an *interpretation*. The method works on the coefficients, and it isn't for nought that the term "synthetic division" is used.

The slope of a curve Δf / Δx contains the notion of division (or ratio). See also the definition of *tangent* in trigonometry for a right-angled triangle. This notion of a slope generates the link between derivative and integral, as the integral uses (Δf / Δx) * Δx to find Δf. The fundamental theorem of calculus is: A function gives the area under its derivative. [12]

A crucial insight:

> When we want to find the root factor x + 1 in (x^2 – 1), then we don't have to assume $x \neq$ -1, but we can assume the unrelated $x \neq$ 1, and then isolate the root factor as (x + 1) = (x^2 – 1) / (x – 1).

One might deem this acceptable. It might be a rationale for the theory of "rational function" – group theory version (RF-GT) to define such singularities away. This theory doesn't seem to care that we must also say something about factor x + 1 at x = 1, but it would be straightforward to plug those holes in multiplicative form. The key question then is:

> If we are willing to assume $x \neq$ 1 and adjust the domain afterwards (in multiplicative form) to again include it, then why would we not do so for $x \neq$ -1 directly ?

Reasoning like this generates the notion of the dynamic quotient as a useful extension of our vocabulary.

Students must simplify algebraic expression like (x^2 – 1) / (x – 1) anyhow. Since the dynamic quotient allows them to do so consistently with (x^2 – 1) // (x – 1), there is no reason not to allow them to do so for the derivative too.

Eliminating factors is one way of simplification. There might be more ways. Thus the dynamic quotient uses the general notion of "simplification".

[11] https://en.wikipedia.org/wiki/Horner's_method#Description_of_the_algorithm
[12] http://mathworld.wolfram.com/FundamentalTheoremsofCalculus.html

3.4. Perspective on division

The core of the new algebraic approach to the derivative lies in a new look at division. While division is normally defined for numbers, we now use the extension with variables and expressions with variables. Variables have their domains. By default the domain is the real numbers. (There might be symbols with unspecified (only potential) domains though: the "nondetermineds" of RF-GT.) Thus, while Descartes, Fermat, Newton and Leibniz didn't have Cantor's set theory, we now use this to replace a bit of Analysis by Algebra. (Instead as happens in the theory of rational functions, that Cantor's set theory is used to widen the gap.)

Let us distinguish the passive division result (noun) from the active division process (verb). For didactics it is important to write y for the numerator and x for the denominator, and not the other way around. In the active mode of dividing y by x we may first simplify algebraically under the assumption that $x \neq 0$, or that 0 is not in the domain of the denominator. Subsequently the result can also be declared valid for $x = 0$. This means *extending the domain*, i.e. not setting $x = 0$ but merely including that element in the domain.

Active division is not an entirely new concept since we find the main element of simplification well-defined in the function Simplify in *Mathematica,* see Wolfram (1996). For us there is the particular application of Simplify[y / x]. This doesn't claim that this well-definedness satisfies conditions for RM. For empirical research, it removes ambiguity, where students will have various levels of skills on simplification, and we can refer to the computer output as an empirical standard. The active notion of division still requires a separate notation for our purposes. Denote it as y // x or ($y\ x^D$) where the brackets in the latter notation are required to keep y and x together, and where the D stands for *dynamic division*. In the same line of thinking it will be useful to choose static $H = -1$, and have $x\ .\ x^H = 1$ for $x \neq 0$. H gives a half turn as imaginary number i gives a quarter turn.

There is already an active notion (verb) in taking a ratio $y : x$. But a ratio is not defined for $x = 0$. Normally we tend to regard division y / x as already defined for the passive result without simplification – i.e. defined except for $x = 0$. Non-mathematicians will tend to take y / x as an active process already (so they might denote the passive result as y // x instead). For some it might not matter much, since we might continue to write y / x and allow both interpretations depending upon context. This is what Gray & Tall (1994) call the "procept", i.e. the use of both concept and process: "The ambiguity of notation allows the successful thinker the flexibility in thought (...)". In that way the paradoxes of division by zero are actually explained, i.e. by confusion of perspectives. It seems better to distinguish y / x and y // x.

3.5. Already used in mathematics education

Clearly, mathematics education already takes account of these aspects in some fashion. In early excercises pupils are allowed to divide 2 a / a = 2 without always having to specify that a must be nonzero. At a certain stage though the conditions are enforced more strictly. A suggestion that follows from the present discussion is that this process towards more strictness can be smoother by the distinction between / and //.

An expression like $(1 - x^2) / (1 - x)$ is undefined at $x = 1$ but the natural tendency is to simplify to $1 + x$ and not to include a note that there is branching at $x \neq 1$, since there is nothing in the context that suggests that we would need to be so pedantic, see **Table 2**, left column. This natural use is supported by the right column. The current practice in teaching and math exams is to use the division y / x as a hidden code that must be cracked to find where $x = 0$, but it should rather be the reverse, i.e. that such undefined points must be explicitly provided if those values are germane to the discussion. Standard graphical routines also tend to skip the undefined point, requiring us to give the special point if we really want a discontinuity.

Table 2. Simplification and continuity

Traditional definition overload	*With the dynamic quotient*
$f(x) = (1 - x^2) / (1 - x) = 1 + x \quad (x \neq 1)$ $f(1) = 2$	$(1 - x^2) // (1 - x) = 1 + x$

In common life there is no need to be very strict about always writing “//”. Once the idea is clear, we might simply keep on writing “/” as a procept indeed. It remains to be tested in education however whether students can grow sensitive to the context or whether it is necessary to *always* impose strictness. For the mathematically inclined pupils or students graduating at highschool one would obviously require that they are aware that y / x is undefined for $x = 0$ and that they can find such points.

3.6. Subtleties

The classic example of the inappropriateness of division by zero is the equation

$$(x - x)\,(x + x) = x^2 - x^2 = (x - x)\,x,$$

where unguarded "division" by $(x - x)$ would cause $x + x = x$ or $2 = 1$.

This is also a good example for the clarification that the rule, that we should never divide by zero, actually means that we must distinguish between:

- *creation* of a quotient by the choice of the *infix* between $(x - x)\,(x + x)$ and $(x - x)$
- *handling* of a quotient such as $(x - x)\,(x + x)$ *infix* $(x - x)$ once it has been created.

The first can be the great sin that creates such nonsense as 2 = 1, the second is only the application of the rules of algebra. In this case, $x - x$ is a constant (0) and not a variable, so that simplification generates a value Indeterminate, for both infices / and //. (One may notice that $x - x = 0$ is the zero polynomial $Q[z] = 0$ in the reference to RF-GT above.)

Also $(a\,(x + x) / a)$ would generate $2x$ for $a \neq 0$ and be undefined for $a = 0$. However, the expression $(a\,(x + x) // a)$ gives $2x$, and this result would also hold for $a = 0$, even while it then is possible to choose $a = x - x = 0$ afterwards: since then it is an instant (and not presented as a variable).

3.7. The derivative

The algebraic definition of the derivative then follows directly:

$$f'[x] = \{\Delta f // \Delta x, \text{ then set } \Delta x = 0\}$$

This means first algebraically simplifying the difference quotient, expanding the domain of Δx with 0, and then setting Δx to zero.

The Weierstraß $\varepsilon > 0$ and $\delta > 0$ and its Cauchy shorthand $\lim(\Delta x \to 0)\ \Delta f / \Delta x$ are paradoxical since those exclude the zero values that are precisely the values of interest at the point where the limit is taken. Instead, using $\Delta f // \Delta x$ on the formula and then extending the domain with $\Delta x = 0$, and subsequently setting $\Delta x = 0$ is not paradoxical at all. Students only need an explanation why one would take those steps.

Much of calculus might well do without the limit idea and it could be advantageous to see calculus as part of algebra rather than a separate subject. This is not just a didactic observation but an essential refoundation of calculus. E.g. the derivative of $|x|$ traditionally is undefined at $x = 0$ but would algebraically be sign[x], see Colignatus (2011b). The derivative gives the change in the area under the curve, and this might not be the same as the slope of the incline (tangent line).

3.8. Differentials

There is the following progress from 2011 to 2016:

- COTP (2011ab) uses "***df* / *dx***" as a icon only, or "***d* / *dx***" as an operator, to link up with history only, so that everyone who still uses this notation for the derivative can see that this has the same outcome,
- Colignatus (2016d) proposes to use dx and dy as variables, and to define $dy = f'[x]\, dx$ so that dy // $dx = f'[x]\, dx$ // $dx = f'[x]$. This is actually the situation with the ray that this section started out with. Thus the derivative $f'[x]$ is found by other means, and then is used to set up the ray with dx and dy. The dynamic quotient dy // dx should not be confused with finding the derivative (since dy is defined by using the derivative).

For users new to the notions of the dynamic quotient and the algebraic approach to the derivative, the relation dy // $dx = f'[x]$ might be confusing since they might think that the dynamic quotient suffices to find the derivative, without the need to set $\Delta x = 0$. (An answer to this is: There are various roads to Rome but only few ways to build it. Check again what the proper definition of the derivative is.)

3.9. Derivative at a point $x = a$

In the standard notation for the derivative, x is fixed and the new variable is Δx.

There is also a notation when x is retained as a variable, and the fixed value is $x = a$. If we want to find the derivative at a point $x = a$ then we would use above method to find $f'[x]$ and then substitute the value to find $f'[a]$. This suffices.

If one wishes to specify a in the deduction, then use:

$$\{ (f[x] - f[a]) \;//\; (x - a), \text{ then set } x - a = 0 \} = f'[a]$$

The following notation would be advised against, since it mixes changes of perspectives:

$$\{\Delta f \;//\; \Delta x, \text{ then set } \Delta x = x - a = 0\} = f'[a]$$

NB. An angle on didactics:

The form $(f[x] - f[a])$ // $(x - a)$ may be more agreeable to students than $(f[x + \Delta x] - f[x])$ // Δx. An intermediate solution is already to use $(f[x + h] - f[x])$ // h. There is $(f[b] - f[x])$ // $(b - x)$, that after resolution still can be evaluated at $x = b$ to find the derivative $f'[x]$ and then be evaluated at $x = a$. However, the use of Δx relates to "difference" and "***df* / *dx***" and has the advantage that there is no subtraction $b - x$. Potentially, it might be better didactics to present the notations alongside to each other, which the invitation to students to use the notation that they like, so that they are encouraged to look deeper into this.

4. Definition of the incline

I will use the word "incline" instead of "tangent (line)" since the incline may also cut the function, see Colignatus (2016e). Let us use "tangent" in trigonometry only.

The core notion is that the slope s must be taken as the *slope of the curve* at the point of consideration. We don't have just the line. First we determine the slope of the curve, and then create the incline with it.

The point-slope form with Δx is: $y - f[x] = s\, \Delta x$ at the point of inclination $\{x, f[x]\}$.

The point-slope form with a is: $y = \text{incline}[x] = s\,(x - a) + f[a]$ at the point of inclination $\{a, f[a]\}$.

The standard form is $y = c + s\, x$, with slope is s and constant $c = f[a] - s\, a$.

5. Basic question, also for readers of WIC Prelude

Since $f[x] - f[a] = 0$ for $x = a$, and if f is polynomial, we have $x = a$ as a root and $x - a$ as a factor. For a flat incline including extrema, or $f'[x] = 0$, we could refer to Suzuki (2005), and the work of Jan Hudde in the years 1657-1658. If the slope of the function is nonzero (as geometry might suggest), then we know that the trigonometric tangent $(f[x] - f[a]) / (x - a)$ must have a nonzero solution at $x = a$, though we cannot find it by substituting $x = a$ directly. But we can use the property that the slope of the function is nonzero.

> When we have $f[x] - f[a] = q[x] (x - a)$ for an unknown factor $q[x]$, then we can isolate $q[x]$ by assuming $q[x] \neq 0$ for $x = a$, and find the $q[x]$ making $(f[x] - f[a]) / q[x] = x - a$ also for $x = a$. We haven't divided by zero and find $(f[x] - f[a]) / (x - a)$ to be basically equivalent to $q[x]$ except that the first is undefined at $x = a$ and the second not.

Having found $q[x]$ we can consider $q[a]$ too. Given the definition of the trigonometric tangent this must be the slope of the function and hence we take it as the slope of the incline.

This method is actually an application of the dynamic quotient, as $(f[x] - f[a]) // q[x] = x - a$, since we are assuming that $q[x]$ is nonzero.

A question is: If one would accept this, then why not directly $(f[x] - f[a]) // (x - a) = q[x]$?

This also avoids a complication when $q[a] = 0$, for a horizontal tangent.

PM 1. See **Appendix H** on the hidden asymmetry in $a\ b = c$.

PM 2. **Appendices B or C** use the error[x], that has a factor $q[x] - q[a]$ so that there is *a* double root. However, above $f[x] - f[a]$ is only part of this error.

The dynamic quotient provides for an algebraic approach to the derivative. WIC Prelude then is relevant to see how far one can come without the dynamic quotient.

6. WIC Prelude's redefinition of "tangent" (closer look)

A secant may cut a curve at more points. The line $y = -x$ cuts x^3 in $x = 0$ only once, but turning the line around shows three points for $y = x$. For the incline, these points would overlap. Range turns this into a definition, see Figure 6. Range (2014:388) states that this redefinition of "tangent" would be known by "algebraic geometers" so that this would not be his initiative, but I have not looked at references here, see the section on *History* below.

Figure 6. Range (2016c:5)

Definition 2.1. *A tangent to a curve at the point P is a line that intersects the curve at that point with multiplicity two or higher, that is, a suitable arbitrarily small rotation of the line around P will separate P into two or more points of intersection.*

Now that we have a more precise definition of a tangent we can look for appropriate tools to identify such tangents, i.e., to find lines that intersect the curve with multiplicity two or higher. The introduction of *coordi-*

I don't want to seem impolite, but it really is a good exercise, namely to create some distance from this text, to replace "tangent" by "orange tree", and thus define "orange tree" as having a double root. Later in Prelude:32, here **Appendix A** Figure 11, there is proof of a double root iff $q[a]$ = s. Hence, this must be the slope of the "orange tree". I have reproduced this proof also Colignatus (2016g) under "Range's proof method". My problem is that there is no link

between "orange tree" and the notion of *slope of the function itself.* This is given by Range (2011:415) but not in the Prelude. Thus there is need for **Appendices B or C**.

Actually, I find Range (2014:389) quite charming. Namely rearrange:

$$f[x] - f[a] = q[x]\ (x - a)$$

into

$$f[x] - (f[a] + q[a]\ (x - a)) = (q[x] - q[a])\ (x - a)$$

and then recognise:

(a) $y = f[a] + q[a]\ (x - a)$ is a line with slope $q[a]$.
(b) $f[x] - f[a] = 0$ generates the root for this equation.
(c) $q[x] - q[a] = 0$ generates the double root for the new LHS.
(d) $q[x] = (f[x] - f[a]) / (x - a)$ for $x \neq a$.
(e) though uncharmingly in the Prelude with unstated link to the slope of the function itself.

Let us discuss this in steps. Observe about this definition of the tangent:

(1) This definition of "tangent" looks at (geometric) points and not (algebraic) roots, for which Range provides a later redefinition.
(2) The above indicates the need for a search to find a suitable rotation, yet, this does not appear to be required for the formal deduction.
(3) The formal deduction in algebra turns the multiplicity of *points of intersection* into the notion of multiplicity of roots *not for the curve itself* but for the *error function,* see also Range (2016c:p32), here **Appendix A**, Corrollary 7.2:

$$\text{error} = \varepsilon[x] = f[x] - \text{incline}[x]$$

It appears that Range *uses* the error function but does not state *explicitly* that this can be recognised as an error term. (He uses the term "remainder" but it takes some algebra to show that remainder and error would be the same.) Note that the geometry of the error is more complicated than just two points on a curve. Perhaps Range wishes not to draw attention to this complexity.
Range's assumption might be that it is *obvious* that when we look at both line and function that there is an error involved, yet, this is a textbook, and this issue must be elucidated. Students are really helped by a diagram on the error to help them to focus of the different concepts involved. (There are so many obvious issues that have actually been explained to us.)
(4) For students there will be a potential confusion between multiplicity of roots of the function $f[x]$ and the multiplicity of roots for the error function $\varepsilon[x]$. The distinction may dawn on them when the error is used consistently (even when not talked about). However, for polynomials, the error is also a polynomial, and perhaps students need more time to grow aware what specific polynomial they are investigating. See below.
(5) (PM. For consistency: Multiplicity of roots is a notion from algebra. Why not use the full scale of algebraic notions, like also the "dynamic quotient" ? Check **Appendix E**.)
(6) (PM. Is the notion of "algebraic functions" not too restricted, in a bit artificial manner ? By giving a name to something it becomes an expression, e.g. Exp[*x*]. Expressions can be handled algebraically again. See how computer algebra handles expressions.)
(7) Use of the error function is obviously similar to the common introduction for the standard approach with limits, with the secant line approaching the incline, see Colignatus (2016d) referring to standard texts and diagrams by Spiegel 1962 [13] and Adams & Essex 2013. [14]

[13] https://boycottholland.files.wordpress.com/2016/12/1962-murrayspiegel-p58-fig4-1.jpg?w=545&h=554
[14] https://boycottholland.files.wordpress.com/2016/12/2013-03-adams-calculus-acompletecourse-p236-figure.png

[15] Whether *two points overlap* or *two lines overlap* doesn't seem to make a material difference, especially when two points define a line. Yet, as said, the geometry of the error is more complicated than just two points on a curve (e.g. a third point). (Basically, Range would agree with this, since at the end of the Prelude he discusses 2^*x*, and presents a diagram that reminds of (but is not fully) the secant diagram.)

(8) **Fundamentally:** In the traditional definition of the incline, there is explicit reference to the *slope of the curve,* and such explicit reference is entirely missing from above definition. **Thus, thus there will a need to clarify what the relation of this "tangent" will be to the *slope of the curve*.** (See Appendices B or C.)

(9) A bit curious: I recently proposed the word "incline" for what is commonly called the tangent. The notion of "touching" namely is not relevant since the incline may also cut the curve. The word "incline" was chosen because it refers to the inclination, or *slope of the curve*, which is the key notion that is relevant here. However, Range links up to the notion of "tangency" as reflected in the notion of "touching" (two points overlapping). In his exposition the slope of the function has no prime position. Thus the difference in perspectives is already reflected in the choice of words.

(10) Below, Range will show that the *slope* s *of his "tangent"* will be equal to some expression $q[a]$ that has been calculated from error $\varepsilon[x] = 0$. **However, he does not explain that this *q*[*a*] is the *slope of the curve.*** We have only two overlapping points and a line through them. (See Appendices B or C.)

Thus, crucially:

- *In the vocabulary of both the limit approach and the algebraic approach by COTP,* the incline is defined as the line that adopts the slope of the curve at the point of interest. The expression "find the incline (tangent)" indirectly asks "find the slope of the curve".
- *In the vocabulary of Range (2016c),* the "tangent" is defined as the line where two intersecting points overlap. When he states "find the tangent" then this generates such a lline, but we are still in the dark what this means for the *slope of the curve*.
- It appears that we cannot really find this link between "tangent" and "slope of the curve" in the pages of Range (2016c) (the Prelude) so that an essential part of didactics on the derivative is missing. The reader of WIC Prelude might assume that this would be given in the full book (2016a) but we aren't looking there. We can find a statement however in Range (2011). Thus **Appendix B** repairs the problem by providing the theorem that is missing in Range (2016c), WIC Prelude. (This present comparison concerns the algebraic approach to the derivative, and isn't a review of WIC itself. For Range's views, we might interview him, but we are currently looking at the presentations in the different publications.)
- The reader is advised to take the Prelude and search for the word "slope". In most cases it would be necessary to replace "slope of the tangent" by "slope of the curve", except in cases where it is clarified that *s* is the slope of the incline (and must be determined by establishing the slope of the curve).

Thus, also Range (2014:288) (bottom right) comes into a new light:

> "Next, the direct algebraic approach to derivatives avoids the introduction of deep new concepts involving limits and continuity early on in a context where—as we just saw—they clearly are not necessary and may even cause confusion. If the major part of a first course in calculus ends up focusing on the mechanical aspects of differentiation anyway, primarily involving algebraic functions, shouldn't it help the students to be able to do all that without having to worry about limits?"

This quote only survives because of the redefinition of tangent from trigonometry into tangent by "touching", or by replacing the incline by the double root line. The referee of the AMS Notices should have been more alert on requiring that this be put in clear terms. Reviewer Ruane (2016) doesn't mention the issue either.

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[15] Or wikipedia, a portal no source: https://en.wikipedia.org/wiki/Derivative#/media/File:Lim-secant.svg

7. Tedious factorisation and switch to rules of diffferentiation

The focus of the algebraic approach to the derivative is *not* on tedious rules of factorisation. Shen & Lin (2014:13), also quoted in Colignatus (2017c):

> "Calculating the slope using the factorization method works for polynomial functions, but the procedure is tedious."

I agree with this. When composing COTP, I used factorisation only for constant, line and square, and proceeded quickly with the standard rules of calculus, by proving them (recursively) for degree *n.*

Range (2016bc) may feel the same about this. The setting of "algebraic functions" is used to determine the rules of differentiation (addition, multiplication, ratio, chain). Actual applications of factorisations are limited to some key examples and some exercises. Students will also see that it quickly becomes tedious, whence there is advantage in learning the rules.

While WIC Prelude develops the rules for "algebraic functions", the main body of WIC thus must repeat (as he explains in the "Notes for instructors") the deduction for the general case, using limits. Some students might enjoy the repetition, for others it might show that mathematicians are mere lawyers of space and number who stick to the roadmap and must provide evidence for every step taken.

COTP introduces the rules of differentiation while using the dynamic quotient, using polynomials as the example. Thus, when it is shown later that exponential functions and trigonometry fit the dynamic quotient (also Colignatus (2017b)), then the same rules apply.

8. WIC Prelude's didactic steps

8.1. Polynomial in general

Let us check the steps in the WIC Prelude. First there is the factorisation of polynomials. It appears that this aspect can be better understood within the context of the theory of "rational functions" (RF). The discussion of this has been moved to Colignatus (2017d). A summary was already given above. **Appendix H** was included for this purpose too.

8.2. Error without name

Subsequently in Figure 7, there are the *points of intersection*, with a formulation of an error but no clarification that it is an *error function*. That is, the name "P.5" is not informative. As said, one may consider it obvious that line and function create an error of approximation, but for students this must be explained.

Figure 7. Range (2016c:17-18)

We are now ready to apply the double point method to an arbitrary polynomial P, which we might as well assume to have degree ≥ 2. We fix a point $(a, P(a))$ on its graph. A non-vertical line through this point has equation $y = P(a) + m(x - a)$, and its points of intersection with the graph of P are the solutions of the equation

$$P(x) - [P(a) + m(x - a)] = 0. \tag{P.5}$$

We need to find the slope m so that this equation has a zero of multiplicity at least 2 at $x = a$. Since $P(x) - P(a)$ has a zero at a, Proposition 5.1 implies that $P(x) - P(a) = q(x)(x - a)$, and similarly it then follows that $q(x) - q(a) = k(x)(x - a)$, where q and k are polynomials of appropriate degrees. We want to emphasize that the polynomials q and k that are determined by these factorizations depend also on the point a that has been fixed. By combining the two factorizations one obtains

$$\begin{aligned} P(x) - [P(a) + m(x - a)] &= q(x)(x - a) - m(x - a) = [q(x) - m](x - a) \\ &= [q(a) - m](x - a) + [q(x) - q(a)](x - a) \\ &= [q(a) - m](x - a) + k(x)(x - a)^2. \end{aligned}$$

This representation shows that the equation (P.5) has a zero of multiplicity at least 2 at a if and only if $m = q(a)$.

We are thus justified in making the following definition that is just an algebraic version of the earlier geometric Definition 2.1.

Definition 5.3. *The **tangent line** to the graph of a polynomial P at the point $(a, P(a))$ is the (unique) line through $(a, P(a))$ that intersects the graph at that point with multiplicity at least 2. The slope of the tangent is called the **derivative** of P at a, and it is denoted by $D(P)(a)$, or also by*

Note that it is actually a key theorem: "root of multiplicity at least 2 at *a* if and only if *m* = *q*[*a*]."

See **Appendices B or C** for the argument that we need to embed this theorem into the proper conditions, that this *q*[*a*] actually represents the *slope of the curve* and not just a plain number from another polynomial.

8.3. Relation to the slope of the curve

The relation to the *slope of the curve* is entirely missing, here in the Prelude.

Check Definition 5.3:

- Definition 5.3 refers to the *slope of the tangent,* but not to the *slope of the curve.*
- Readers should be warned that they might read texts about the "tangent" as if there is reference to the *slope of the curve*. However, WIC Prelude redefines "tangent" to read as "two points of intersection" that are overlapping, whence this doesn't refer to a *slope of the curve*.
- There is the derivation that *m* = *q*[*a*] but there is no explanation that *q*[a] would be the *slope of the curve.* We only know that *q*[*x*] is a factor, no more.

8.4. Late and incomplete discussion of the slope of the curve

There is only a late discussion of what the *slope of the curve* is.

- On page 13 there is the first appearance of the *difference quotient*. It is employed for *average velocity* = Δ*distance* / Δ*time* (P.2). However, it is not related to the notion of a slope. PM. On page 15 the discussion is burdened with notions of approximation and continuity, which aren't relevant here because we are still in the realm of polynomials. [16] (Apparently, the Prelude rather prepares for the subsequent chapters.)
- On page 38 for exponential functions we finally see *q*[*x*] = Δ*f* / Δ*x* for Δ*x* ≠ 0, though not explicitly yet with the delta's.
- Page 39 employs the reference on page 15 to notions of approximation and continuity to now argue that exponential functions can be treated in like manner. This is convoluted. It wasn't necessary on page 15. If it is necessary on page 38, then use the case of page 38. (Not intended to be unkind: Don't create confusion on earlier pages with the objective to allow for confusion on later pages.)
- Page 39 has the remarkable statement: "(...) that the unknown slope *m* of the *tangent* can be approximated by *q*[*x*] (...)", see Figure 8. Truth is, that we are interested in the *slope of the curve,* and that we use this to create the incline. Range writes about finding the *slope of the tangent* as if this were some independent notion of itself.
- Page 40 finally has the standard graph of the secant.
  (a) **Finally an explicit statement about a slope**: "slope *q*(*x*) = (2^*x* − 1)/*x* for *x* > 0."
  (b) Yet, the incline (tangent line) is still missing in the diagram, see Figure 9

Figure 8. Range (2016c:39)

for the value $q(0)$. The geometric version of this idea in the present setting suggests that the missing value $q(0)$ for the slope of the tangent should be approximated by the slope of lines through $(0, 1)$ and a second *distinct* nearby point $(x, 2^x)$ on the graph as $x \neq 0$ approaches 0. (See Figure 9.) In fact, for $x \neq 0$, the slope of such a line is given precisely by the quotient $q(x)$.

It certainly looks very plausible that the unknown slope m of the *tangent* can be approximated by $q(x)$ as the *non-zero* value of x gets closer and closer to 0. In Figure 9, as $x > 0$ moves closer and closer to 0, the point

[16] With some didactics on the side: There are 5 pages to discuss instantaneous velocity, compared to 2 pages in COTP (2011:151-152).

Figure 9. Range (2016c:40)

40 *What is Calculus? From Simple Algebra to Deep Analysis*

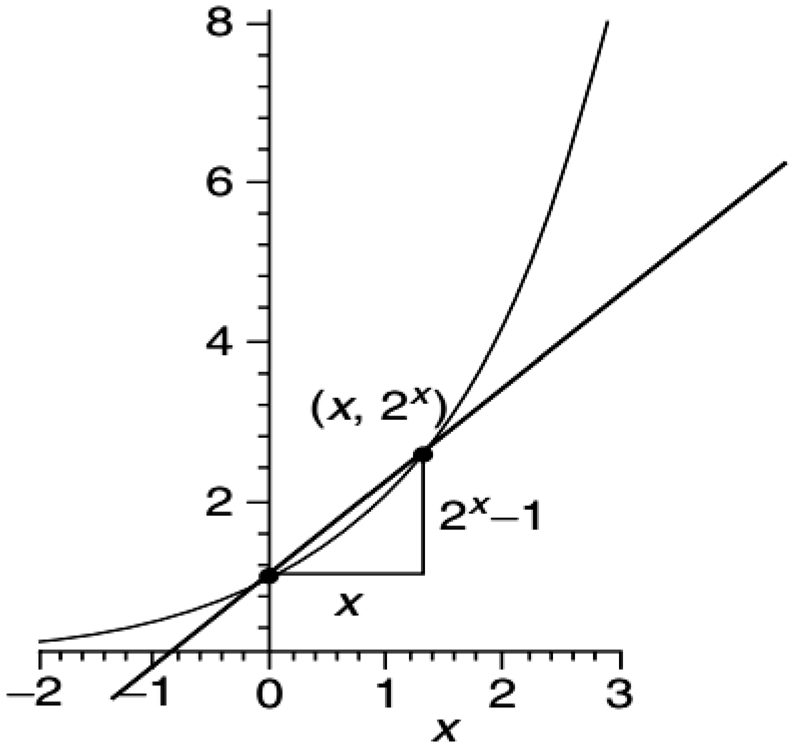

Fig. 9 Secant to $y = 2^x$ of slope $q(x) = (2^x - 1)/x$ for $x > 0$.

8.5. Comparison with Range (2011:415)

WIC Prelude or Range (2016c) thus is in stark contrast with the explicit statement in Range (2011:415), see Figure 10.

Figure 10. Range (2011:415)

Note that translating the equation

$$f(x) - f(a) = q(x)(x - a)$$

into

$$\frac{f(x) - f(a)}{x - a} = q(x) \text{ for } x \neq a$$

provides the interpretation of the factor $q(x)$ as a rate of change for $x \neq a$. Since q is a priori known to be continuous at a, the definition $f'(a) = q(a)$ shows that the derivative is well approximated by (average) rates of change. This is the crucial fact

I haven't checked whether the latter is stated in such manner in later sections of Range (2016a) though I presume that it is. Such later statements might be seen as correcting for the

omission in the Prelude (2016c). However, the slope of the curve should be mentioned in the Prelude, and it isn't sufficient to later "correct" the absence there.

8.6. (2011:415) is a missing link in (2016c)

Range (2016a) has a proposal for didactics, and directly executes this proposal. He doesn't seem to claim that he presents an essential refoundation of calculus. COTP has the latter claim, yet presents the development in the form of a primer since its focus is on didactics and not mathematics. [17]

Range (2016c)'s assumption in the Prelude is that for education it would be advantageous to develop this intuition of a root with a multiplicity higher than 1, whence he redefines "tangency" as having a double root, whence questions on (this) "tangency" can be answered by finding that root.

My problem with this approach to didactics is that the notion of slope (Range (2011:415)) disappears both from the intuitive phase and from major proofs.

- In this *other* publication, Range (2011:415) agrees that $s[x, a] = \Delta f / \Delta x = q[x]$ for $x \neq a$ is a slope.
- In this formula on the LHS $s[x, a] = \Delta f / \Delta x$ is the slope of a linear expression for x fixed too (and $s = s[a, a]$).
- In this formula on the RHS $q[x]$ would generate the slope $q[a]$ of the function f at $x = a$. Range (2016c) does not show this for the *Prelude*. His statements are that the "tangent has a slope" and not that the "curve has a slope".

The missing link is provided in **Appendices B or C.**

NB. On continuity, mentioned in Figure 10, some remarks can be made here – but see also below:

Also observe that Range (2011:415) explicitly refers to the continuity of $q[x]$, while Prelude Lemma 7.1 (**Appendix A**) holds that q is an algebraic function (and apparently thus must be continuous).

Also, in Range (2016a) there is of course the Front Matter (2016b), and there on page xxvi there is Caratheodory's definition of differentiability at a, with a factorisation *that must be continuous* at a. This actually doesn't differ really from above difference quotient in (2011:415). The point is that this notion is somewhat hidden in the Prelude, and it destroys the claim that limits are avoided (since continuity apparently is defined with limits).

PM 1. The WIC Front Matter arguments on page xxvii for preferring Caratheodory's definition have been discussed here implicitly. Caratheodory's definition is not preferable for highschool since this definition of differentiability does not focus on the slope. (We should develop a curriculum such that students know about continuity from junior highschool, and they would not be put-off by Caratheodory's reference to continuity. But the present issue is the slope.)

PM 2. The implied suggestion that continuity would be handled by the reference to rational functions, is rejected by Colignatus (2017d). We may accept that this theory implicitly manipulates the domain, but the foundation for it is an implied reference to Analysis with continuity and limits on numbers, while the algebraic approach in COTP uses the algebraic expression as sufficient. (Notwithstanding that one might see group theory as an effort by algebraists to show that numbers are algebra too.)

[17] The AMS Notices "Booklist" rejected inclusion of COTP because it has the form of a textbook, thus neglecting the point that it presents an essential refoundation of calculus. However Colignatus (2009, 2015) "Elegance with Substance" has been included, that includes the discussion from ALOE.

8.7. Intermediate conclusions

(1) On content, Range is aware that an incline can also cut a curve (say x^3 at x = 0). Yet in his didactics he has expounded on the notion of tangency as "touching". Together with the criterion of two roots, this mix caused him to provide a new definition of "tangency", that focuses on finding the solution with two roots. This caused a vocabulary, with questions like "find the slope of the tangent", which phrase has been borrowed from another context, but that now starts a life of its own. This vocabulary now actually is separated from the focus on the *slope of the curve* itself, since the Prelude-trained student has a focus on finding a double root.

(2) *The Prelude does not show how the property of a double root is linked to the slope of the curve.* The slope of the curve however is the key issue for didactics of the derivative.

(3) This can be repaired, given Range (2011) (2014). Still, it took time for me to grow aware of what the problem actually was, then to design the proper proof structure, and then polish this up. The result is in **Appendix B or C**.

9. Using Appendices B or C. The condition of continuity

Appendix B uses the derivatie at $x = a$ and **Appendix C** uses the derivative at x with a new variable called Δx. This generates the following points, and let me continue the numbering:

(4) On content, if we accept *factorisation* (and ways to identify factors by other means than division) and *continuity* of the found factors, then the condition of a double root indeed is equivalent to finding the slope of the curve and using the standard incline. This is calculation only. Thus, the numbers that are found are the same. Thus, on content, Range's vocabulary in (1) still appears to be consistent with finding the derivative (though it is confusing when you don't know about **Appendix B**).

(5) WIC Prelude or Range (2016c) also provides some cases of *factorisation.* However, this exposition is less strong w.r.t. the available methods for identifying factors. See Colignatus (2016f) for some methods: (i) coefficients, (ii) Ruffini's Rule, (iii) division by nonzero factors, (iv) potentially more ways in the algebra of expressions (e.g. long division on formulas).

(6) However the Prelude's treatment of *continuity* on page 34, reproduced in **Appendix D**, is curious.

(6a) Prelude theorem 7.3 in Range (2016c:34) or **Appendix D** states that factorisation implies that all "algebraic functions" are continuous. It is presented as a *consequence.* However, the theorem in **Appendices B or C** requires it as a *condition*. If we want to interprete the calculated value for the double root line as the slope of the function itself, then we need this. Thus the Prelude can find the a double root line, but this requires continuity (as proven in **Appendix D**) to find the slope of the function. (One might hold that the continuity exists even when not proven, but then this wouldn't be mathematics anymore.)

(6b) His approach to continuity appears to rely on a notion of a limit. If he thinks that he needs proof of continuity, then **the claim that limits are avoided is false.** They are only **not presented,** but apparently still **required on content.** (The promise at the beginning of the Prelude that ideas like tangent can be introduced without limits, is only fulfilled because the Prelude changes the definition of "tangent".)

(6c) The intuition at the very beginning of the overlapping intersections doesn't elaborate on the need for continuity, and thus one wonders about the intuition. If the intuition is not so strong that it provides an answer to this issue on continuity, can it be used as a foundation really ? Would it not be better to return to the *intuition of the slope of the curve* directly ? A line has a slope, so why would a curve not have a slope at some point ?

(6d) Range refers to the definition of differentially given by Caratheodory in all cited works, except for the Prelude. This definition contains the condition of continuity. This should settle the question whether Range requires continuity for the trigonometric tangent or incline: he does. Only the double root line doesn't need it. For polynomials it can be found by Ruffini's Rule.

10. On approximation

The dynamic quotient is an algebraic notion, and the derivative can be found by purely algebraic means. The expression contains information, and this is used for formal continuity instead of numerical continuity. The need for this algebraic approach already shows in the line, when we want to recover the slope $s = (y - c) / x$ also at $x = 0$. With the tool of the dynamic quotient we can also handle curves.

The trigonometric tangent is conventionally defined as $\Delta y / \Delta x$ for nonzero denominator. We should be aware that this is a definition for numbers. When we switch to algebra with variables with domains, then the proper form becomes Δy // Δx. The reasoning in algebra is that it looks at the form of the expression and not the numbers. The expression Δy // Δx has the form of a slope just like $\Delta y / \Delta x$.

When we consider the geometry of the situation, then we can define the incline (tangent line), and then there arises the notion of approximation and error. Not the other way around. It is true that ALOE and COTP (in the back) provide additional theoretical justification that the geometry of the error supports the algebraic notion of a slope of the function at the point of interest, yet, this should not create confusion about the direction of the reasoning. There is no need for a limit precisely because there is no limiting process. Any value can be substituted, and for a particular substitution the error happens to be zero.

On the other hand WIC Prelude first tends to hide approximation and error, in the introductory phase of "touching", and then gradually gives it pride of place as the only serious game in town. One might see this as the didactics of a warm towards a cold or hot shower, depending upon how one appreciates the role of limits for calculus.

But the main issue here is to separate the notion of derivative from the notion of limit anyhow.

In the AMS Notices Range (2014:387) (right column) states:

> "Everyone now agrees that the limit is 2*a*; the result that is obtained by plugging in $x = a$."

Range apparently doesn't see that the issue remains as simple as this. Earlier developers Descartes, Ferman, Newton and Leibniz didn't have the notions of sets and domains, and thus one can understand that they didn't come up with the notion of the manipulation of the domain.

Range (2014) continues:

> "Of course the instructor warns that this result, i.e., Lim[$x \to a$, $x + a$] = 2*a,* requires a proof, since we can’t just use $x = a$ in a formula that was derived under the assumption that $x \neq a$. This apparently so simple matter is really quite nontrivial, and it caused a lot of difficulties already in the seventeenth century as calculus was being developed."

Given the history of mankind and calculus the issue is nontrivial indeed. Yet, common belief nowadays is that 1 + 1 = 2 is trivial, and we best start regarding the algebraic approach to calculus in likewise fashion.

I agree with the notion of proof here, otherwise it wouldn't be mathematics. Students should grow aware when simplification and plugging are allowed. However, the proof would neither be the limit as Range suggests, but would be found in the definition of the dynamic quotient. This definition itself provides direct proof for the derivative in algebraic fashion.

Why not accept and use a definition that is helpful for students ? The use of limits for the derivative is diagnosed as an employment project for teachers, reminiscent of the USSR. There is really no need for it, *really, really, really.*

The expression "Lim[$x \to a$, $x + a$] = $2a$" might be a private joke that even such a simple expression can also be subjected to such treatment, or it might be a reference to the theory of rational functions, with the equivalence class for {$x + a$, $(x^2 - a^2) / (x - a)$, ...}. This is reasoning from theory and not from empirically observing students.

11. On limits and numerical continuity of functions

We can, see Colignatus (2016bc):

- *define continuity with limits*, as Lim[$x \to a$, $f[x]$] = $f[a]$
- or *define limits with continuity*. [18]

Textbooks commonly present both approaches simultaneously, and do not care about the circularity (since they might also merely want to show that both approaches are possible). Obviously students can become confused because of this circularity (don't see this).

The derivative causes logical order. Differentiability implies continuity. Thus, we better start with the continuum, use this to define continuous functions, and then develop the derivative with it. That is, *if* we rely on a numerical approach.

Range (2016c)'s exclusion of limits and reliance on continuity seems to fit that order. However:

(1) For the first pages of the "Prelude to Calculus" the student has to rely on an intuitive grasp of *numerical* continuity of functions.
(2) When we repair the omission on the *slope of the curve* by providing **Appendices B or C**, then there appears to be an explicit reliance on continuity of $q[x]$ at $x = a$. This would still be okay if we can develop a notion of continuity without the use of limits. (See the intermezzo that the limit would rely on this notion of continuity.)
(3) Range (2016c:34), reproduced in **Appendix D**, provides a formal approach to continuity, but this relies on a notion of a limit (here written in multiplicative form so that also zero is possible). The claim that limits are avoided is false. The only true claim is that *the didactics in Range (2016c) rely on an intuition of (numerical) continuity and limits without a formal development of limits*.

I only look at Range (2011, 2014, 2016bc) and not the full book Range (2016a). Range states that continuity is looked at in more detail in the subsequent parts. Yet, this still creates the tension that the student needs the more complex theory to support the assumed original intuition.

There is still the alternative approach in COTP. It is better to use the algebraic notion that expressions contain information about the functions. The algebraic approach to the derivative in COTP uses the dynamic quotient that simplifies expressions. There is no need to speak about such a formal definition of a limit, since the approach is not based upon numerical considerations. Yet, if one would wish to do so, then one can simplify $q[x]$ into such a $u[x]$.

12. Evaluation of approaches in didactics

Colignatus relies on the notion of "algebraic simplification". This is partly equivalent to Range's reference to factorisation, since factors are easy to simplify. The difference in approach is that Range switches to limits to find the derivatives for exponential functions and trigonomety, while Colignatus shows that there are algebraic solutions too. (Colignatus

[18] For example: $q[x]$ has limit L at $x = a$ if we can find a continuous function $u[x]$ such that $q[x] = u[x]$ everywhere for $x \neq a$, and then $L = u[a]$.

(2017b) summarises the case for the exponential functions.) Potentially there are even more ways for "algebraic simplification".

Two points of agreement:

- Both authors accept that factorisation is possible for what Range calls the "algebraic functions", WIC Prelude or Range (2016c) p32, Lemma 7.1, reproduced in **Appendix A**.
- Both authors accept that *if factorisation with continuity is possible*, then the existence of a double root in the error term is equivalent to the existence of a (numerically) *continuous* factor $q[x]$ and a factorisation $f[x] = q[x] (x – a)$ such that $q[a]$ gives the slope of $f[x]$ at the point $x = a$. This requires Range (2011:415) which is not in WIC Prelude, and what is stated more explicitly in **Appendices B or C**.

The two authors disagree about the didactic relevance of the latter givens. Range finds the two givens relevant to use them as a *Prelude to Calculus*. Colignatus wasn't aware of these givens when COTP was written, but rejects them now as didactically relevant, since the derivative concerns the slope of the function, and the relation to the double root distracts and is only simple sometimes and generally tedious (or with the need to teach Ruffini's Rule that only works for polynomials).

Obviously, these latter two positions are only hypotheses w.r.t. the empirics of didactics, and they only help to focus on strict research questions. It are the students who have to show (in randomised controlled experiments) what works for them.

13. History

I am new to the approach as suggested by Range on polynomials and rational functions. Much thus is tentative. His presentation makes me curious about the history of the approach.

Suzuki (2005) clarifies that early developers of calculus were aware of choices involved in the algebraic or numerical approaches. For polynomials the rules could be derived and they sufficed (Jan Hudde in 1657-1658, 360 years ago). Working with polynomials directly quickly becomes tedious. The approach with limits had the clear advantage that it worked for more than polynomials. (Apparently they didn't see the approaches on exponential functions and trigonometry in COTP, or those were found and lost again.) In the past there likely wasn't a clear distinction between on one side *mathematics* as a body of theory and on the other side *didactics of mathematics* as an empirical science w.r.t. teaching and learning. Still, there was an awareness of the main choices involved.

Thus my impression is that the criticism expressed above w.r.t. the approach indicated by Range is not really new, but would fit similar considerations already in the past, if one would research history on it. Range provides some references to the past, like with Descartes or the definition by Caratheodory with factoring.

It would be interesting when a historian of mathematics and its didactics would cast some light on the considerations in the past, and whether this current round of discussion only surfaced since we have forgotten about those.

14. Conclusions

Most conclusions have been moved to the *Introduction.*

Standard Analysis and WIC Prelude are essentially the same. Compared to this, COTP provides a real alternative, also in algebra, that deals with expressions. The dynamic quotient provides both better didactics and an essential refoundation of calculus.

Yet, the criterion of double root applies, and when it causes neater deductions than this can be used, as long as it is clear that this is only a manner of calculation and not the fundamental concept for the definition of the derivative.

Appendix A. WIC "Prelude to Calculus", p32 (key proof)

Figure 11. Range (2016c:32)

$\mathcal{A}$. Functions in $\mathcal{A}$ are also called *algebraic*.

The most important fact is that the familiar factorization result for polynomials generalizes to functions $f \in \mathcal{A}$, as follows.

Lemma 7.1. ***(Factorization Lemma.)*** *If $f \in \mathcal{A}$ and a is in the domain of f, then there exists $q \in \mathcal{A}$ defined on the domain of f such that*

$$f(x) - f(a) = q(x)(x - a). \tag{P.12}$$

The proof of this statement basically involves checking through the proofs of the rules we discussed in the preceding section, where in each instance we were able to conclude that, given the factorization for the initial functions, one ends up with an appropriate factorization of the function that results by application of one or several of the admissible operations.

Based on the factorization result, it is clear how to generalize the notion of multiplicity of a zero a to the case of a function $f \in \mathcal{A}$.

In analogy to the case of polynomial and rational functions, successive application of the factorization lemma then implies the following result.

Corollary 7.2. *Given $f \in \mathcal{A}$ and the factorization (P.12), then*

$$f(x) - [f(a) + m(x - a)] = (q(a) - m)(x - a) + k(x)(x - a)^2$$

for some other $k \in \mathcal{A}$ that is defined on the domain of f.

Geometrically, this means that the line described by the linear function $y = f(a) + m(x - a)$ intersects the graph of $y = f(x)$ at $(a, f(a))$ with multiplicity at least two if and only if $m = q(a)$. Consequently, the line given by $y = f(a) + q(a)(x - a)$ is *the tangent* to the graph of f at $(a, f(a))$. This shows that the function $f \in \mathcal{A}$ is algebraically differentiable at a, with derivative $D(f)(a) = f\,'(a) = q(a)$, where q is defined by (P.12).

Appendix B. Theorem on double root line and slope of the curve (x – a)

Theorem: If factoring is possible and such a factor is continuous at $x = a$ then:

$((\exists\, q[x])$ & ($q[a]$ the slope of $f[x]$ at $x = a$) & $(s = q[a])) \Leftrightarrow (\exists$ double root $x = a$ for $\varepsilon[x])$

Step	*Rewriting of the error ε*	*Deductions on the incline*
0	First steps are needed to understand the terms of the sub-theorem in step 13	
1		Incline taken at point $\{a, f[a]\}$. At that point $(x – a) = 0$ and $f[x] – f[a] = 0$
2		Definition: y = incline$[x] = s\,(x – a) + f[a]$
3		PM. Slope incline: [19] $s = (y – f[a]) / (x – a)$
4	Definition: $\varepsilon[x] = f[x]$ – incline$[x]$	$\varepsilon[x] = f[x] – y$
5	$\varepsilon[x] = f[x] – f[a] – s\,(x – a)$	If $\varepsilon[x] = 0$ and $x – a = 0$ then $f[x] – f[a] = 0$
6		From 5: $s = ((f[x] – f[a]) – \varepsilon[x]) / (x – a)$ $(x \neq a)$
7		$f[x] – f[a] = 0$ thus it has root $x = a$
8		**Assume factor:** $f[x] – f[a] = q[x]\,(x – a)$
9		Assuming $q[x] \neq 0$: [20] find $q[x]$ such that $(f[x] – f[a]) / q[x] = (x – a)$ avoiding division by $x – a$
10		Secant slope: $q[x] = (f[x] – f[a]) / (x – a)$ $(x \neq a)$
11		$q[a]$ slope [21] of $f[x]$ at $x = a$ (i.e. *a* number too)
12	$\varepsilon[x] = (q[x] – s)\,(x – a)$	
13 sub	$(\exists\, q[x]$ & ($q[a]$ the slope of $f[x]$ at $x = a$) & $s = q[a]) \Leftrightarrow (\exists$ double root $x = a$ for $\varepsilon[x])$	
14 ($\Leftarrow$)		There is a double root $x = a$ for error $\varepsilon[x]$
15		Include steps 5-12 to find 12
16		Double root: $(q[x] – s)$ must be 0 at $x = a$ too
17		$q[a] – s = 0$ or $s = q[a]$ ($\Leftarrow$ proven)
18 ($\Rightarrow$)		There is $s = q[a]$ with $q[a]$ slope of $f[x]$ at $x = a$
19	$\varepsilon[x] = (q[x] – q[a])\,(x – a)$	Include 5-12 for meaning that $q[a]$ is a slope
20		$q[x] – q[a] = 0$ thus it has *a* root $x = a$
21		**Assume factor:** $q[x] – q[a] = k[x]\,(x – a)$
22		Assuming $k[x] \neq 0$: find $k[x]$ such that $(q[x] – q[a]) / k[x] = (x – a)$ avoiding division by $x – a$
23	$\varepsilon[x] = k[x]\,(x – a)$^2	Thus there are two roots. ($\Rightarrow$ proven)
24 PM1		From 6 or 12: $s = q[x] – \varepsilon[x] / (x – a)$ $(x \neq a)$
25		Thus: [22] $s = q[x] – k[x]\,(x – a)$ (factoring)
26 PM2	$\varepsilon[x] = (q[a] – s)\,(x – a) + Rem[x]$	Remainder: [23] $Rem[x] = (q[x] – q[a])\,(x – a)$
27		NB: $Rem[x] = \varepsilon[x]$

Now there is a clear link how the condition of *a* double root generates the *slope of the curve*.

[19] It is even the slope at $x = a$ because of the definition.
[20] $f[x] – f[a]$ is only part of the error $\varepsilon[x]$, and only $\varepsilon[x]$ has *a* double root. Horizontal inclines with $q[a] = 0$ are excluded from finding the factor in this manner, but see the next footnote.
[21] Range (2011:415) assuming Caratheodory (continuity); or COTP (form of expression).
[22] The division by one root factor must leave another so that the error or remainder is zero.
[23] Range (2011:406) ($q[a] = s$ iff double root), but this doesn't give the slope (steps 10 & 11).

Appendix C. Theorem on double root line and slope of the curve (Δx)

Theorem: If factoring is possible and such a factor is continuous at $\Delta x = 0$ then:

$((\exists\, q[x]) \,\&\, (q[x] \text{ the slope of } f[x]) \,\&\, (s = q[x])) \Leftrightarrow (\exists \text{ double root } \Delta x \text{ for } \varepsilon[\Delta x])$

PM. For comparising with **Appendix B**: variable $\Delta x = x - a$ or $x = a + \Delta x$.
Thus rewrite (there "x" as here "$x + \Delta x$") and (there "a" as here "x").

Step	*Rewriting of the error ε*	*Deductions on the incline*
0	First steps are needed to understand the terms of the sub-theorem in step 13	
1	PM. Δx links up with COTP:224.	Incline taken at point $\{x, f[x]\}$. At that point $\Delta x = 0$ and $\Delta f = f[x + \Delta x] - f[x] = 0$
2		Definition: $y = \text{incline}[x, \Delta x] = s\, \Delta x + f[x]$
3		PM. Slope incline: [24] $s = (y - f[x]) / \Delta x$
4	Definition: $\varepsilon[x, \Delta x] = f[x + \Delta x] - \text{incline}[x, \Delta x]$	Notation for ease: $\varepsilon[\Delta x] = \varepsilon[x, \Delta x]$
5	$\varepsilon[\Delta x] = \Delta f - s\, \Delta x$	If $\varepsilon[\Delta x] = 0$ and $\Delta x = 0$ then $\Delta f = 0$
6		From 5: $s = (\Delta f - \varepsilon[\Delta x]) / \Delta x \quad (\Delta x \neq 0)$
7		$\Delta f = 0$ thus it has root Δx
8		**Assume factor:** $\Delta f = q[x + \Delta x]\, \Delta x$
9		Assuming $q[x + \Delta x] \neq 0$: [25] find $q[x]$ such that $\Delta f / q[x + \Delta x] = \Delta x$ avoiding division by Δx
10		Secant slope: $q[x + \Delta x] = \Delta f / \Delta x \quad (\Delta x \neq 0)$
11		$q[x]$ slope [26] of $f[x + \Delta x]$ at $\Delta x = 0$ (i.e. a number too)
12	$\varepsilon[\Delta x] = (q[x + \Delta x] - s)\, \Delta x$	
13 sub	$(\exists\, q[x] \,\&\, (q[x] \text{ the slope of } f[x]) \,\&\, s = q[x]) \Leftrightarrow (\exists \text{ double root } \Delta x \text{ for } \varepsilon[\Delta x])$	
14 ($\Leftarrow$)		There is a double root Δx for error $\varepsilon[\Delta x]$
15		Include steps 5-12 to find 12
16		Double root: $(q[x] - s)$ must be 0 at $\Delta x = 0$ too
17		$q[x] - s = 0$ or $s = q[x]$ ($\Leftarrow$ proven)
18 ($\Rightarrow$)		There is $s = q[x]$ with $q[x]$ slope of $f[x]$.
19	$\varepsilon[\Delta x] = (q[x + \Delta x] - q[x])\, \Delta x$	Include 5-12 for meaning that $q[x]$ is a slope
20	$\varepsilon[\Delta x] = \Delta q\, \Delta x$	$\Delta q = 0$ thus it has a root $\Delta x = 0$
21		**Assume factor:** $\Delta q = k[x, \Delta x]\, \Delta x = k[\Delta x]\, \Delta x$
22		Assuming $k[\Delta x] \neq 0$: find $k[\Delta x]$ such that $\Delta q / k[\Delta x] = \Delta x$ avoiding division by Δx
23	$\varepsilon[\Delta x] = k[\Delta x]\, (\Delta x)^2$	Thus there are two roots. ($\Rightarrow$ proven)
24 PM		From 6 or 12: $s = q[x + \Delta x] - \varepsilon[\Delta x] / \Delta x \quad (\Delta x \neq 0)$
25		Thus: [27] $s = q[x + \Delta x] - k[\Delta x]\, \Delta x$ (factoring)

[24] It is even the slope at $\Delta x = 0$ because of the definition.
[25] $f[x + \Delta x] - f[x]$ is only part of the error $\varepsilon[\Delta x]$, and only $\varepsilon[\Delta x]$ has *a* double root. Horizontal inclines with $q[x] = 0$ are excluded from finding the factor in this manner, but see the next footnote.
[26] Range (2011:415) assuming Caratheodory (continuity); or COTP (form of expression).
[27] The division by one root factor must leave another so that the error or remainder is zero.

Appendix D. Range (2016c:34) on continuity

It is not entirely clear why Range wants to show that factorisation of *f*[*x*] – *f*[*a*] implies that *f*[*x*] is continuous at *a.* For this present text, I did not read Range (2016a) beyond Range (2016bc).

We conclude the discussion of algebraic functions with another important consequence of the factorization (P.12).

Theorem 7.3. *Given $f \in \mathcal{A}$ and a point a in the domain of f, there exist numbers $\delta > 0$ and K, such that one has the estimate*

$$|f(x) - f(a)| \leq K\,|x - a| \quad \text{for all } x \text{ with } |x - a| < \delta. \tag{P.13}$$

We had seen the significance of this kind of estimate already in Section 4, where it was used to recognize that the instantaneous velocity $v(t_0)$ is well approximated by average velocities over shorter and shorter time intervals around t_0. The crucial property expressed by the estimate (P.13) is that the values $f(x)$ approach $f(a)$ as $x \to a$, since clearly the left side of (P.13) becomes increasingly smaller as $|x - a| \to 0$. This is the essence of what is known as the *continuity* of the function f, a fundamental property that will be discussed more in detail in Chapter II. As we shall see in the next section, this approximation property is the critical ingredient that will allow us to study the tangent problem for more general functions that are not of algebraic type.

Proof. The proof of the theorem easily follows from the fact that functions $q \in \mathcal{A}$ are *locally bounded*, as follows: given a in the domain of q, there exist numbers $\delta > 0$ and K that depend on q and a, so that

$$|q(x)| \leq K \text{ for all } x \text{ with } |x - a| < \delta. \tag{P.14}$$

In order to prove the estimate (P.13), recall that by (P.12) one has $f(x) - f(a) = q(x)(x - a)$, where $q \in \mathcal{A}$ as well. Now use the above local bound (P.14) for the factor q to obtain

$$|f(x) - f(a)| = |q(x)|\,|x - a| \leq K\,|x - a|$$

for all x with $|x - a| < \delta$. ■

Appendix E. Range (2016c:13-14) actually uses the dynamic quotient

Michael Range (2016:13-14) actually *uses* the steps for the "dynamic quotient" in COTP (2011:57) but without reference to its formal development. This is no surprise, since the dynamic quotient has been developed to capture the practice in mathematics, see Colignatus (2016bc). Yet, the major conceptual step for mathematicians is to accept the definition of the dynamic quotient, as difficult as it can be to accept a definition. (Notably: it is important to grow aware that the formula for a function also contains key information.)

Check the steps: (i) Assume that the denominator is nonzero, (ii) Deduce an outcome, (iii) then define the missing point with the outcome. This would apply to any quotient, not only for the derivative, and thus would be relevant for "pre-calculus" courses in junior highschool. It is only a consequence for the higher grades that once you have the dynamic quotient, then the algebraic approach to the derivative follows directly. Thus the key question for me is whether Range would be willing to turn what he is doing here into a general algebraic procedure.

by formula (P.2), since this formula now gives the meaningless expression $\frac{0}{0}$. However, if we rewrite the equation that defines velocity as the product *distance* $=$ *velocity* $\times$ *time*, then the problem becomes more manageable. In fact, let us consider the simple case considered by Galileo, i.e., $d(t) = ct^2$. If we fix a particular time t_0, then $d(t) - d(t_0) = ct^2 - ct_0^2$, which factors into

$$d(t) - d(t_0) = c(t + t_0)(t - t_0). \qquad \text{(P.3)}$$

Note that if $t > t_0$ the factor $c(t + t_0)$ in this last formula obviously equals the average velocity over the time interval from t_0 to t. (Just divide both

velocity is allowed to be both positive and negative (or zero), with the sign accounting for the direction of motion along a line. More generally, when the motion is not constrained to a line, the velocity is represented by a so-called *vector,* a more complicated quantity that encodes, for example, the direction of the motion in space.

sides of (P.3) by $t - t_0 \neq 0$.) This also holds if $t < t_0$, where the time interval now goes from t to t_0. (See Problem 2 of Exercise 4.1.) Therefore, trusting in the consistency of the formula (P.3), we are led to define the velocity at t_0 by taking the value of this factor at $t = t_0$, i.e., we define

$$v(t_0) = c(t_0 + t_0) = 2ct_0.$$

Perhaps you have some doubts about the validity of this definition. After

PM. Readers interested in the continuation of this quote may look in **Appendix I**.

Actually, Range (2016b:xvi) here Figure 1 and (2011:404) here Figure 12 acknowledges that there is a perspective of "plugging in" the value of $x = a$ where the limit is taken. My suggestion is to listen better to students, and to wonder why they have this perspective. My interpretation is that students work with formulas and not with numbers (or the definition of a function as pairs of {*x, f*[*x*]}). This algebraic perspective leads to the definition of the dynamic quotient.

Figure 12. Range (2011:404)

1. INTRODUCTION. Have you ever wondered why we burden our students with limits when teaching about tangent lines and differentiation of rational, root, and similar algebraic functions? Of course, as experienced mathematicians, we know that limits ultimately cannot be avoided. However, the early emphasis on limits in the context of differentiation of numerous algebraic examples may cause quite a bit of confusion. After all, the calculation of such derivatives relies primarily on algebraic techniques to rewrite the difference quotient in such a way that one can cancel the troublesome $h = \Delta x \neq 0$ from the denominator. The final answer then follows by what appears to be just *plugging in* $h = 0$. Surely most students, when first shown the derivative of $y = x^2$, are hard pressed to understand the subtlety of the statement $\lim_{h\to 0}(2x + h) = 2x$, a conceptual leap that took mathematicians close to two centuries to fully understand and to formulate correctly. Yes, we try to teach our students that we do need to take the *limit* as $h \to 0$, rather than just *evaluate* at $h = 0$. On the other hand, evaluating at $h = 0$ is eventually justified by invoking the *continuity* of the relevant functions. No wonder today's students in a standard first calculus course typically retain little about limits. They have grown up with graphing calculators, and continuity—at least its intuitive geometric interpretation—looks obvious to them. Consequently it is difficult for them to grasp the need for limits as long as one considers only algebraic functions.

Thinking about these difficulties in the teaching of elementary calculus, I was somewhat surprised to discover that there is a very simple and natural algebraic approach to differentiation of algebraic functions that avoids limits altogether and justifies the students' "easy" calculation of derivatives by "plugging in." More surprising was the realization that this approach had not been used systematically in the early days of calculus, when mathematicians struggled unsuccessfully for nearly two centuries to resolve the inconsistencies and mysteries regarding infinitely small quantities, infinitesimals, and differentials that are zero or nonzero depending on what suits the purpose.[1] While the basic idea already appears in the work of René Descartes

Appendix F. Derivative of f / g

The issue here concerns the derivative of *f*[*x*] / *g*[*x*] for polynomials, and should not be confused with the role of the notion of a rational function in Δ*f* / Δ*x* where *f* is a polynomial. The latter is discussed in the section "Linking up to school mathematics" and **Appendix G**.

Range (2016c:20), the WIC Prelude, refers to "rational functions". The rational functions are defined such that the denominator is never zero, see Colignatus (2017d). [28] If the denominator would be zero then it would not be a rational function. Range takes care to emphasize that *Q*[*a*] ≠ 0. The deduction then is straightforward.

> The algebraic differentiation process that we just discussed for polynomials extends immediately to *rational* functions $R = P/Q$ (i.e., quotients of polynomials) at any point a where R is defined, that is, where the denominator Q is non-zero. Given that $Q(a) \neq 0$, if $R(a) = 0$, then one must have $P(a) = 0$ as well, and hence $P(x) = q_P(x)(x-a)$ for some polynomial q_P. Consequently $R(x) = q(x)(x-a)$, where $q = q_P/Q$ is a rational function defined at a. If $R(a) \neq 0$, one obtains a corresponding factorization
>
> $$R(x) - R(a) = q_R(x)(x-a) \qquad \text{(P.7)}$$
>
> with another *rational* function q_R defined at a. In analogy to the case of polynomials we say that a rational function R has a *zero at a of multiplicity $\geq m$*, where m is a positive integer, if $R(x) = k_m(x)(x-a)^m$ for some rational function $k_m(x)$ defined at a. By an argument analogous to the one used earlier for polynomials, it follows that a rational function R is *algebraically differentiable* at every point a where it is defined, i.e., its graph has a (unique) tangent line at the point $(a, R(a))$ defined by the property

Part of my problem with this is: A person who still wonders about the possibility that *Q*[*a*] = 0 runs the risk of the answer by theorists that those cases are pathological and do not generate neat theorems. However, such cases are really relevant for the algebraic approach to calculus. Notably *Q*[*x*] = (*x* – *a*) clearly has a zero value when *x* = *a,* and the explanations in the theory of "rational functions" that one abstracts from the domain or neglects "removable singularities" might make for nice algebra but would be risky for our purposes.

Appendix G. Retraction

Colignatus (2016efg) were written when Joost Hulshof (Amsterdam) in autumn 2016 claimed that the derivative might be found for polynomials without limits, and Peter Harremoës (2016) informed me about his use of Horner's Method (here: Ruffini's Rule), whence I eventually found the work by John Suzuki (2005), Michael Range (op. cit.) and Samuel Shen & Qun Lin (2014). I have been new to this approach and thus had to digest it.

Looking at Descartes's approach with circles, Colignatus (2016g) found a general product *f*[*x*] – *f*[*a*] = *q*[*x, a, u*] (*x* – *a*), which is not a factorisation because it only holds at *x* = *a,* with *x* = *u* the center of the circle on the horizontal axis. This is used in (2017b) but doesn't generate news. I wondered whether the double root emphasized by Range derived from the phenomenon that there was a (hidden) circle behind it. I now retract that suggestion for polynomials, since the double root for polynomials clearly arises from the phenomenon of polynomials that when *p*[*x*] – *p*[a] = 0 at *x* = *a,* then this can be factored as *p*[*x*] – *p*[a] = *q*[*x*] (*x* – *a*), and then a repeat for *q*[*x*] = *q*[*a*] at *x* = *a.* This would hold for algebraic functions in

[28] https://www.encyclopediaofmath.org//index.php?title=Rational_function&oldid=17805

general. [29] Thus also for Sqrt[*x*] we would rely on its algebraic nature. The suggestion for a role for circles still stands for non-algebraic (transcendental) functions (though I only looked at the WIC Prelude and don't know whether he uses or implies the double root there).

While $f[x] = x^3$ has an incline at $x = 0$ that cuts the curve (rather than touching it), I rather used the spline $f[x] = \{\text{If } x < 0 \text{ then } -x^2 \text{ else } x^2\}$, since it seems that this always cuts the circle somewhere. This line of enquiry is no longer relevant because circles are no longer relevant.

PM. It turned out that Hulshof didn't fully write out his claim and when I asked and received an elaboration then it appeared that he made the same steps as Range: (i) that it is possible to find the *value* of the derivative, (ii) but to also neglect that there is still proof required that this value concerns the *derivative* (because the derivative is defined by the slope of the curve, which still involves a ratio).

Appendix H. An introductory view on group theory from programming

We might look at group theory from the viewpoint of programming. The following programming example uses *input* → *output*. This is asymmetric, since the output isn't necessarily the same as the input. However, let us use = instead of →. This helps to clarify a potential source of confusion.

Steps in programming

Let us suppose that the computer programme knows what *plus* is, and what a *variable* (storage location) is. Consider a repeated addition of a variable and storing the outcome into a new variable.

Computer screen	*Programmer's view*
$a + a + a = c$	This is just addition.

An inverse operation:

$c - a - a - a = 0$	Subtraction, repeatedly, given the above.

We may extend the programme with another feature, called "multiplication", with "factors":

$a + a + a = 3\ a = c$	Introduction of a recorder for the number of repeats in the addition.

The operation calls also for an inverse operation, called "division", and "factor" iff "divisor":

$c / 3 = a$	Introduction of a recorder of the number of repeats in the subtraction. Or a feature to eliminate the number of repeats in addition.

The number of repeats in the addition can be replaced by a variable too:

$a\ b = c$	The factors *a* and *b* give *c.*

Key step: (*) seems like an equivalent statement

$a\ b = c$	(*) *a* and *b* are factors or divisors of *c.*

[29] http://mathworld.wolfram.com/AlgebraicFunction.html

With two substatements (**):

$c / a = b$	(**) a is a factor or divisor of c.
$c / b = a$	(**) b is a factor or divisor of c.

Discussion of these steps

(*) seems innocent when the process is read from input to output. If a and b generate c then one might say that a and b are factors of c. Even when $a = 0$ or $b = 0$ then the outcome $c = 0$ doesn't make it wrong to say that a times b generated c.

The problem emerges in (**), when the order of the process is reversed. The input now consists of both c (numerator) and a particular factor (denominator). The output should be the other factor. If the denominator is zero, then any value in the outcome might be possible, and c should be zero too. (There is no way how c can be nonzero and still be produced by a zero factor and a nonzero factor.) The programmer hasn't introduced this condition yet.

Thus the statement "The factors a and b give c" is *not* equivalent to "a and b are factors or divisors of c".

It is proper to say that *divisors are nonzero, a divisor is a factor, and only a nonzero factor is a divisor*.

It is elementary school stuff, yet, when a group theory version of the rational functions (GT-RF) eliminates the domain and neglects the removable singularities, then there might still arise the confusion that all factors are also divisors. (This confusion may happen when this is translated to proper functions with domains.)

PM. The above gives the rule: If $c \neq 0$ and a and b are factors with $a\ b = c$, then $a \neq 0$, and we can write $c / a = b$ without problem. However, in the main body of this paper, we are looking at $c = f[x] - f[a] = 0$, whence this rule doesn't help us, even though we know that $x - a$ is a factor. We only know that if $f[x] - f[a] \neq 0$, then $x - a \neq 0$, and then we can determine $q[x] = (f[x] - f[a]) / (x - a)$. Our problem however is what happens when $x = a$. The mathematical theory of Analysis states that we need limits for this. Colignatus (2011ab) shows that we only need a theory of algebraic expressions and the ability to manipulate the domain.

Hidden asymmetry (or dynamics) in =

In conventional mathematics the equality sign is symmetric. Thus $a = b$ iff $b = a$.

There is symmetry in arithmetic for 3 0 = 0. For algebra, the above shows that there is a hidden asymmetry, for then we have variables with domains.

When 3 0 = 0 is read left to right (LTR) it is true that the input 3 0 generates output 0. Read right to left (RTL) the input 0 actually generates: For all x, $0 = x\ 0$. Perhaps we can write this as: 0 = {any x} 0. It is partially true that $x = 3$ is one of the possible solutions, yet it isn't the only one, and thus in algebra 3 0 = 0 doesn't give the full truth and symmetry. In algebra 0 = 3 0 becomes an incomplete statement of all possible solutions for solving $0 = x\ 0$. [30]

In RF-GT – see Figure 3 – there is apparently the intention (if I understand this well) to interprete $a\ b = c$ as "a and b are divisors of c", which then includes the possibility that $c = 0$. This happens where RF-GT considers polynomials only (thus polynomials $p / 1$ as rational functions that have nonzero denominator 1). For this apparent intention, e.g. compare Range

[30] This is not the question "Give the factors of zero". For example, "Give the factors of 12" would generate 12 = 2 6 = 4 3, thus multiple outcomes too. The question is "Solve $0 = x\ 0$" and in this context even better: "Find the factor x such that $0 = x\ 0$".

(2014:389) where the left column has the difference quotient and the right column has the multiplicative format (with "removed singularity"). In that case, the multiplication for polynomials is *interpreted* as division. For $c = 0$ the implied suggestion is that there would be unique combinations of factors *a* and *b,* restricted only by an equivalence class, just like when $c \neq 0$. However, when $c = 0$ and $a = 0$ then any *b* might be possible. This is still a *function* when you interprete this as Div[*c, b*] → *a* for nonzero *b.* But this kind of interpretation also generates *not* a function but a *correspondence* Div[*c, a*] → *b*. Thus the (presumed) interpretation of multiplication as division fails.

In **Appendix I**, Figure 13, Range (2016c:14) actually recognises this for an equation $0 = k\ 0$. See there for a discussion that his argumentation isn't convincing. It is better to grow aware of the distinction between polynomial and "rational function" and the need to adjust the domain.

(For the ratio format, the denominator must be nonzero, but for the interpretation of multiplication of factors as divisors it is not quite clear whether the theory of rational functions imposes such a restriction. If the restriction is used, then there would be no difference with division, so why do so difficult by *interpreting multiplication as divison* (if it is division by nonzero elements anyway) ?)

I am not at home in RF, either RF-FL or RF-GT. My suggestion is that group theory first resolves this hidden asymmetry in its use of the equality sign. Let one adopt a notation such that equality is symmetric, and confusion is avoided. It seems okay that we use this hidden asymmetry in elementary school, since we teach pupils not to divide by zero. But when group theory creates the impression that it interpretes $a\ b = c$ such that factors are seen as divisors, then RF-GT doesn't adopt that rule anymore, and then it better be put into the notation.

The notational problem might be resolved by using $a\ b = c$ only for nonzero factors and use $a\ b \to 0$ if there is a zero factor. I am blank about the option whether this would actually also be better for elementary school. (Would kids understand $3\ 0 \neq 0$ but $3\ 0 \to 0$? Three times zero reduces to zero, and isn't quite equal to zero. For, $0 \to x\ 0$ for any number *x*.)

There is a key difference between numbers and variables that have domains. My suggestion is that the dynamic quotient likely provides the required notation also for group theory. The current RF-GT would survive as a theory of expressions, but not for functions with domains. Yet I am only a teacher of mathematics and no research mathematician, and I am not qualified to judge on this, and thus this remains a suggestion only.

PM. We might use the property that polynomials put restrictions on the solution space. For (x^2 – 1) = (x – 1) (x + 1) there are these discussion steps:

- The interpretation (x^2 – 1) / (x + 1) = (x – 1) seems to work when $x \neq -1$ even when $x = 1$, and we have the form 0 / {a particular nonzero, now 2} = 0.
- The interpretation (x^2 – 1) / (x – 1) = (x + 1) seems to work for $x \neq 1$ even when $x = -1$, and we have the form 0 / {a particular nonzero, now –2} = 0.
- These restrictions are actually no different from the earlier rule: *divisors are nonzero, a divisor is a factor, and only a nonzero factor is a divisor*.
- Observe that 0 / {a particular nonzero} = 0 is only partially true if read RTL. Symmetry would require: 0 / {any denominator ≠ 0} = 0. Thus 0 / 2 = 0 and 0 / -2 = 0 use a hidden asymmetry.
- However, the polynomial has restricted the solution space to roots {-1, 1} where factors would be zero. When a particular root *x* is used is used to create the 0 in both numerator and result, then the denominator *must* use an element in the *remaining* solution set, and thus be nonzero.
- In this case the polynomial has two factors and thus {any denominator ≠ 0} reduces to {a particular nonzero}. For more factors, the solution set however would be larger. For polynomials we should write {any denominator ≠ 0 in the remaining solution set}.
- However, we still can use: *c* / {n – 1 factors} = {1 remaining factor}. In that case, it is fair to use {a particular nonzero} as denominator.

- Thus, since polynomials restrict the solution space, we can allow the structure: 0 / {a particular nonzero} = 0 as also a symmetrical expression, however *conditional* on the assumption that the 0 has been created from only 1 remaining factor.

Thus, the correspondence now is replaced by a conditionality. This still deviates from the notion of a function.

It remains that (x^2 – 1) = (x – 1) (x + 1) allows x = -1 and (x^2 – 1) / (x + 1) = (x – 1) requires $x \neq -1$.

These conceptual problems are resolved by (x^2 – 1) // (x + 1) = (x – 1) that has a symmetrical equality sign (without a hidden asymmetry).

Equivalence class

The above on a nonsingle solution set should not be confused with the notion of an equivalence class.

Speaking about equivalence classes, it may be noted that the theory of rationals (Q) declares 1 / 2 and 2 / 4 equivalent, like the theory of rational functions does for x / (2 x) and (2 x) / (4 x). Part of the problem here is that the sign "/" is used both as an operator and for denoting a number. It might again be that group theory facilitates confusions in elementary school. If we denote 1 / 2 as 0.5, then we can reduce 1 / 2 and 2 / 4 to operations and phases in a computation that aren't a final result yet. This doesn't entirely resolve the matter since we must establish that 0.5 and 0.49999... would be equivalent too. Yet, this comment may be an eye-opener that group theory focuses on *existence of numbers* while the crucial question for students and didactics is on *notation* without confusion, see Colignatus (2016h) (2017a).

(Group theorists will object that there are a / b for huge numbers a and b so that it would be humanly impossible to determine the decimal expansion, so that there is value in the notion of an equivalence class. Yet the principle a / b = c is already given by the very operation of division. It aren't actually the operations that are relevant but rather the elements in the set.)

Appendix I. WIC Prelude on what is convincing for the derivative

Appendix E already showed where the Prelude actually uses the steps in the dynamic quotient without actually developing or recognising the formal whole. Our quote stopped at the phrase: "Perhaps you have some doubts about the validity of this definition." COTP explained the validity by referring to the algebra of expressions. There is information within the expressions that can be used to manipulate the domain.

Range first uses that Δ*f* = *q*[*x*] Δ*x* gives a "universal truth" in physics, and that it is reasonable to take this *q*[*x*] as rendering the value of Δ*f* / Δ*x* at Δ*x* = 0. This is something for physics. The second justification gives the interpretation of the incline, when acceleration stops and the object proceeds at constant speed. This is correct, but the reference of "two points in time that just happen to coincide" is distractive, and biased on his definition of "tangent".

Figure 13. Range (2016c:14)

Perhaps you have some doubts about the validity of this definition. After all, the basic formula *distance* = *velocity* × *time* reduces, in the case $t = t_0$, to the equation $0 = c(t_0 + t_0) \cdot 0$, which surely is correct, but then any other number k also satisfies the equation $0 = k \cdot 0$. So you may ask why do we single out the particular number $c(t_0 + t_0)$ among all the other possible numbers k that satisfy the equation?

One justification surely comes from the fact that $c(t_0 + t_0)$ is exactly that number that arises when t is replaced by t_0 in the algebraic formula $d(t) - d(t_0) = c(t + t_0)(t - t_0)$. Since this formula does represent a "universal truth", the value of $c(t + t_0)$ at $t = t_0$ should have an interpretation that is analogous to that for all other values t, that is, it should represent a velocity. And since only one moment in time t_0 is involved, it is reasonable to think of $c(t_0 + t_0)$ as the velocity at t_0.

Another justification is based on the geometric interpretation involving tangents to parabolas that we discussed earlier in Section 3. As we showed then (just replace $x = t$ and $y = d(t) = ct^2$), the line through the point (t_0, ct_0^2) with slope $2ct_0$ is the tangent to the graph of the function $d(t) = ct^2$, i.e., it is that line that fits the graph in an "optimal" way. Rephrasing this in the context of motion we thus can say that at the moment $t = t_0$, the *constant* speed motion $l(t) = ct_0^2 + 2ct_0(t - t_0)$ with velocity $2ct_0$ (i.e., the equation that defines the tangent) provides an optimal description of the motion given by $d(t) = ct^2$ at that moment. More precisely, this constant speed motion matches the given motion described by $d(t)$ at the moment t_0 "with multiplicity *two*", that is, at two points in time that just happen to coincide. Alternatively, think of a vehicle starting from rest at $t = 0$ under the same uniform acceleration as a falling stone, so that—according to Galileo—the distance traveled at time $t > 0$ equals $d(t) = ct^2$. At time t_0 the driver takes off his foot from the accelerator. Neglecting minor factors such as friction, air resistance, and so on, the car would continue rolling with *constant* velocity equal to $2ct_0$.

Subsequently, Range lays the ground for approximation, continuity and limits, as will be developed in the main body of WIC. Apparently, Range sees this as contributing to a *convincing* interpretation of the derivative. However, the objective of the project was to show that the algebraic approach *suffices,* and that approximation and limits *distract.* Clearly, history shows that the numerical method had some momentum above algebra, and it is definitely useful that the outcomes are the same, but this presentation creates doubts whether Range really regards the algebraic method convincing by itself. Observe also that this is a very simple (quadratic) polynomial, whence it is extra curious that it is suggested that approximation would be required for a convincing case.

Figure 14. Range (2016c:14-15)

Finally we can also consider a *dynamic* point of view, which perhaps reflects most closely the crux of motion with variable speed, as follows. As we saw, for $t \neq t_0$ the value $q(t) = c(t + t_0)$ gives the *average* velocity

during the time interval $[t_0, t]$ (or $[t, t_0]$ if $t < t_0$). Surely we expect that the velocity at t_0, no matter how defined, should be very close to the average velocity over very short time intervals, i.e., when t is very close to t_0, and furthermore, this approximation should improve as the time interval gets shorter, i.e., the closer t gets to t_0. The chosen value $v(t_0) = q(t_0)$ fulfills this expectation perfectly, since

$$|q(t) - q(t_0)| = |c(t + t_0) - 2ct_0| = |c|\,|t - t_0|\,. \tag{P.4}$$

Evidently formula (P.4) shows that when t is "very close" to t_0, then the average velocity $q(t)$ from t_0 to t is "very close" to $q(t_0)$ as well. For example, let us use meters and seconds, so that $c \approx 4.9$ m/sec^2. Suppose $t_0 = 5$ sec and $t = t_0 + 1/1000 = 5.001$ sec; then the average velocity $q(t)$ during the interval $[t_0, t]$ equals 4.9×10.001 m/sec, which differs from the velocity $q(5) = v(5) = 2 \times 4.9 \times 5$ m/sec by $4.9 \times 1/1000 = 0.0049$ m/sec. Stated differently, formula (P.4) gives a precise meaning to the intuitive statement that as t approximates t_0 (we write $t \to t_0$), then $q(t) \to q(t_0)$ as well. As we shall see later, the property we just discussed and that we encode in the statement

$$\text{if } t \to t_0, \text{ then } q(t) \to q(t_0),$$

is an elementary example of a fundamental abstract property that is known as *continuity*.

Appendix J. Table with an overview of issues

Table 3. Overview of some issues in the comparison

	Range (2011) (2014) (2016bc)	Colignatus (2007+)
Scope	This table only looks at "Prelude to calculus" in a much larger book with much larger scope. Not suitable for highschool students and non-math majors	Didactic development of analytic geometry and calculus for highschool and matricola for non-math-majors, primer for teachers, essential redesign of calculus
Limits	The promise that limits are avoided appears true only didactically for the double root line. The chapter already discusses notions of approximation and limits. Limits remain required for the trigonometric tangent (see the reliance on continuity in **Appendices B or C** here). Limits are required on content and *didactically* for exponential functions and trigonometry	Not required for polynomials, exponential functions and trigonometry. Currently required for more. Obviously required for math majors.
Simultaneous introduction of both derivative and integral	Surface is not present in the Prelude. The focus on "touching" blocks attention for trigonometric tangent and area increment	Key notion
One has to start somewhere	Touching of line and circle or curve	Surface under constant function and line
Diagram with slope	Only late on page 40 and without the incline (tangent line). The error term is used but not identified with a name. Students aren't helped in this manner to graps key notions in this issue	Essential reliance on surface while formulas show the simultaneous relevance of derivative and integral
Tangent	Redefined to be the double root line. Teachers used to the standard definition of "tangent" will be confused on the slope of line vs slope of curve	Standard definition of tangent as the line that adopts the slope of the curve
Touching or overlapping with the curve ?	Mention that "tangent" comes from Latin "touching"	Proposal for better name "incline" since the tangent may also cut the curve
Use of double root	It is not dwelled upon that factorisation quickly becomes tedious. Examples tend to be chosen for ease of results. (Some might find Ruffini's Rule easy though, though it isn't mentioned)	Not aware of the method until December 2016. Now, might mention it, but not spend much time on this (simplification only required for one step, then move on to the standard rules)
Handling of area	Isn't discussed	The trick is to assume that *f* gives a surface under some function *g,* and that it is the objective to find this *g.* Thus surface is already given, and we are only interested in the relation between integral and derivative,

		making sure that each step is reversible
Main criticism	(1) There is no proof that the double root line generates the slope of the curve. The Prelude misses the fundamental notion in calculus that we can find the slope of the function. The double root line generates the proper number, but what does it mean ? (2) Why use the double root as introduction to calculus, when the focus is on derivative (slope of the function) and integral (increment, surface) ? (3) Apparently there is some reliance on the group theory notion of "rational function", while this appears to be somewhat problematic in itself too. Requires pre-calculus courses on the polynomial remainder theorem and perhaps Ruffini's Rule	There may be some criticism but it basically turns out then that there is bad reading
Minor points	Repeat of proof of rules for calculus for the different types of functions	See Colignatus (2011b)
Proof of concept	Presented as a proof of concept, but without empirical testing yet. It is not clear in what manner the criticism in the introduction w.r.t. the standard approach is answered exactly. What steps find students crucial improvements for their understanding ? If we test on their understanding of tangency, do we really test on their understanding of tangency ?	Presented as a proof of concept, but without empirical testing yet. COTP focuses on didactics but with only a modest suggestion as to the improvement in didactics. Removing limits should be an objective step towards faster understanding. What works should be determined empirically. However, this developed insight also comes with a claim on a fundamental redesign of calculus,
Major risk	Readers interested in the algebraic approach to calculus may be deterred and lose interest	One grows aware that mathematicians are trained for abstract thought, and that didactics is an empirical science, see Colignatus (2009, 2015)

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Colignatus is the name in science of Thomas Cool, econometrician and teacher of mathematics, in Scheveningen, Holland.

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