# An algebraic approach to the derivative

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**Abstract**

There is a new approach to the derivative, an algebraic one, different from using infinitesimals or limits. The approach originated in research on mathematics education and has been developed to the stage that it can be tested there. For mathematics research there is scope for further development of foundations. The paper shortly reviews the approach and the literature about it.

**Introduction**

Fermat, Newton and Leibniz developed the derivative using infinitesimals. Cauchy and Weierstrasz used limits, while Cauchy used sequences and Weierstraß predicate logic. Weierstraß was acquainted with Cantor and his set theory, but set theory wasn't quite developed yet to be used as follows.

There is a new approach to the derivative that relies on algebra and modern set theory. A key influence is also from computer algebra back to algebra again. The approach originated in research on mathematics education, and has been developed to the stage that it can be tested there. For mathematics research there is scope for further development of foundations. This paper shortly reviews the approach and the literature about it.

**Perspective on division**

The core of the new algebraic approach to the derivative lies in a new look at division. While division is normally defined for numbers, we now use the extension with variables and expressions with variables. Variables have their domains. By default the domain is the real numbers. There might be symbols with unspecified (only potential) domains though.

Let us distinguish the passive division result (noun) from the active division process (verb). For didactics it is important to write $y$ for the numerator and $x$ for the denominator, and not the other way around. In the active mode of dividing $y$ by $x$ we may first simplify algebraically under the assumption that $x \neq 0$, or that 0 is not in the domain of the denominator. Subsequently the result can also be declared valid for $x = 0$. This means *extending the domain*, i.e. not setting $x = 0$ but merely including that element in the domain.

Active division is not an entirely new concept since we find the main element of simplification well-defined in the function Simplify in *Mathematica,* see Wolfram (1996). For us there is the particular application of Simplify[$y$ / $x$]. The active notion of division still requires a separate notation for our purposes. Denote it as $y$ // $x$ or ($y\ x^D$) where the brackets in the latter notation are required to keep $y$ and $x$ together, and where the $D$ stands for *dynamic division*. In the same line of thinking it will be useful to choose static $H$ = -1, and have $x\ .\ x^H = 1$ for $x \neq 0$. $H$ gives a half turn as imaginary number $i$ gives a quarter turn.

There is already an active notion (verb) in taking a ratio $y : x$. But a ratio is not defined for $x = 0$. Normally we tend to regard division $y$ / $x$ as already defined for the passive result without simplification – i.e. defined except for $x = 0$. Non-mathematicians will tend to take $y$ / $x$ as an active process already (so they might denote the passive result as $y$ // $x$ instead). For some it

might not matter much, since we might continue to write *y* / *x* and allow both interpretations depending upon context. This is what Gray & Tall (1994) call the "procept", i.e. the use of both concept and process: "The ambiguity of notation allows the successful thinker the flexibility in thought (...)". In that way the paradoxes of division by zero are actually explained, i.e. by confusion of perspectives.

**Definition of dynamic division**

To make these notions strict, let *y* / *x* be as it is used currently in textbooks, and let *y* // *x* = ($y\ x^D$) be the following process or program, called *dynamic division*:

> *y* // *x* ≡ { *y* / *x,* unless *x* is a variable and then: assume $x \neq 0$, simplify the expression *y* / *x*, declare the result valid also for the domain extension $x = 0$ }.

Simplification only applies when the denominator is a variable but not for numbers. Thus *x* // *x* = 1 but 4 // 0 generates 4 / 0 which is undefined. Also *x* / *x* is standardly undefined for $x = 0$.

This definition assumes a different handling of different parts of the domain. The test on the denominator is a syntactic test. When the denominator is an expression like (*p* + 2) then the syntactic test shows that the denominator is a variable, $x = p + 2$. One does not substitute "(*p* + 2) is a variable" for this doesn't look at syntax but uses the value of the variable.

It has been an option in the {...} definition above to write "(a) variable" instead of "a variable", which allows a shift from the syntactic test towards the semantic test of variability, and which also allows *substitution* into the definition, like "(*p* + 2) is (a) variable". After ample consideration, already in 2007 and later explicitly in Colignatus (2014b), I think that we are better served with the syntactic test on the denominator, since this directly leads to the question: what is the domain of the denominator ?

The use of the curly brackets {...} also borrows from *Mathematica.* The brackets signify a list, that can be a set, but when the elements are expressions then the sequential evaluation of those turns into a programme.

**General application**

Clearly, mathematics education already takes account of these aspects in some fashion. In early excercises pupils are allowed to divide 2 *a* / *a* = 2 without always having to specify that *a* must be nonzero. At a certain stage though the conditions are enforced more strictly. A suggestion that follows from the present discussion is that this process towards more strictness can be smoother by the distinction between / and //.

An expression like $(1 - x^2) / (1 - x)$ is undefined at $x = 1$ but the natural tendency is to simplify to 1 + *x* and not to include a note that there is branching at $x \neq 1$, since there is nothing in the context that suggests that we would need to be so pedantic, see **Table 1**, left column. This natural use is supported by the right column. The current practice in teaching and math exams is to use the division *y* / *x* as a hidden code that must be cracked to find where $x = 0$, but it should rather be the reverse, i.e. that such undefined points must be explicitly provided if those values are germane to the discussion. Standard graphical routines also tend to skip the undefined point, requiring us to give the special point if we really want a discontinuity.

Table 1. Symplification and continuity

| *Traditional definition overload* | *With the dynamic quotient* |
|---|---|
| $f(x) = (1 - x^2) / (1 - x) = 1 + x \quad (x \neq 1)$<br>$f(1) = 2$ | $(1 - x^2) // (1 - x) = 1 + x$ |

In common life there is no need to be very strict about always writing "//". Once the idea is clear, we might simply keep on writing "/" as a procept indeed. It remains to be tested in education however whether students can grow sensitive to the context or whether it is necessary to *always* impose strictness. For the mathematically inclined pupils or students graduating at highschool one would obviously require that they are aware that $y / x$ is undefined for $x = 0$ and that they can find such points.

**Subtleties**

The classic example of the inappropriateness of division by zero is the equation

$$(x - x)\ (x + x) = x^2 – x^2 = (x - x)\ x,$$

where unguarded "division" by $(x - x)$ would cause $x + x = x$ or $2 = 1$.

This is also a good example for the clarification that the rule, that we should never divide by zero, actually means that we must distinguish between:

- *creation* of a fraction by the choice of the *infix* between $(x - x)\ (x + x)$ and $(x - x)$
- *handling* of a fraction such as $(x - x)\ (x + x)$ *infix* $(x - x)$ once it has been created.

The first can be the great sin that creates such nonsense as 2 = 1, the second is only the application of the rules of algebra. In this case, $x - x$ is a constant (0) and not a variable, so that simplification generates a value Indeterminate, for both infices / and //.

Also $a\ (x + x) / a$ would generate $2x$ for $a \neq 0$ and be undefined for $a = 0$. However, the expression $a\ (x + x) // a$ gives $2x$, and this result would also hold for $a = 0$, even while it then is possible to choose $a = x - x = 0$ afterwards: since then it is an instant (and not presented as a variable).

**The derivative**

The derivative then follows directly:

$$f'[x] = \boldsymbol{d}f / \boldsymbol{d}x = \{\Delta f // \Delta x, \text{ then set } \Delta x = 0\}$$

This means first algebraically simplifying the difference quotient, expanding the domain of $\Delta x$ with 0, and then setting $\Delta x$ to zero.

The Weierstraß $\varepsilon > 0$ and $\delta > 0$ and its Cauchy shorthand for the derivative $\lim(\Delta x \to 0)\ \Delta f / \Delta x$ are paradoxical since those exclude the zero values that are precisely the values of interest at the point where the limit is taken. Instead, using $\Delta f // \Delta x$ on the formula and then extending the domain with $\Delta x = 0$, and subsequently setting $\Delta x = 0$ is not paradoxical at all. Students only need an explanation why one would take those steps.

Much of calculus might well do without the limit idea and it could be advantageous to see calculus as part of algebra rather than a separate subject. This is not just a didactic observation but an essential refoundation of calculus. E.g. the derivative of $|x|$ traditionally is undefined at $x = 0$ but would algebraically be sign[$x$]. The derivative gives the change in the area under the curve, and this might not be the same as the slope of the tangent.

**Discussion of the literature of this approach**

The literature on this new algebraic approach is yet small. The approach originated in 2007 in ALOE (1981 unpublished, 2007a, 2011), when I extended a discussion on the logical paradoxes with a discussion of the paradoxes by division by zero. This text was included in EWS (2009a, 2015) with minor polishing up. This discussion is top-down. COTP (2011c) is a bottom-up *proof of*

*concept,* that shows how derivative and integral can be developed in algebraic manner for the functions used in highschool. Relevant are the Reading Notes (2011+) that include comments on COTP. The COTP website contains sheets from a workshop and a (slow) video on Youtube of 2013. Since the algebraic approach was developed for education and concentrates on those conditions, Colignatus (2014b) is a suggestion for research mathematicians on further developing foundations. Richard Gill (2012) of Leiden reviewed EWS and COTP in English in the "Nieuw Archief voor Wiskunde", the journal of the Dutch Royal Society for Mathematics (KWG). Gamboa (2011) reviews the book for the Eur. Math. Soc. and expresses disquieting feelings because of reading familiar material in uncommon format, but once one gets used to this then one can enjoy it. My approach intends to present the logic that students can follow, while traditional presentations have not been designed for didactics. There are two reviews of EWS and COTP in Dutch but curiously those do not present the algebraic approach to the derivative. CWNN (2015h) discusses division for elementary education, using algebraic *H.* Since my work concentrates on education and applied mathematics, research mathematicians will be served by the discussion "What a mathematician might wish to know about my work", which has been included now in the Appendix of the 2nd edition of EWS.

Colignatus (2012d) gives a short comparison with David Tall on the education of the derivative. The algebraic approach with the distinction between passive (noun) and active (verb) division fits, as said, with Tall's distinction between concept and process, and his 'procept' idea. My suggestion is that it is better to distentangle these two perspectives rather than confuse them into a procept. I expected that Tall would appreciate this new analysis, also based upon computer algebra. Instead Tall (2013) and in earlier conversation with me at Dutch ORD 2010 continues to prefer his original approach of Tall & Sheath (1983), that the computer is used graphically to focus in onto slopes, and that students use their fingers to trace the slope of functions. However, ultimately it are the students themselves who must show, in empirical testing, what works best for them.

The development of this new algebraic approach to the derivative has been hindered much by a curious "math war" since about 2005 about mathematics education in Holland, with some fallout for the USA, see Colignatus (2016). This "math war" concerns another topic, namely "realistic mathematics education" (RME) versus "traditional mathematics education" (TME) (where this tradition isn't Euclid). Though this "math war" concerns another topic, it apparently drains attention from new insights, and anyone who doesn't take sides apparently is judged unreliable or assigned to the "enemy". Let other countries be warned about the disastrous consequences of such unscientific behaviour.

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