Sketch of the influence by G. Mannoury (1867-1956) on his students P.M. van Hiele (1909-2010) and A.D. de Groot (1914-2006), and the continued importance of this influence for a general theory of epistemology (2015), didactics of mathematics (2008) and methodology of science (including the humanities) (1990)

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Summary

A discussion on epistemology and its role for mathematics and science is more fruitful when it is made more specific by focussing on didactics of mathematics and methodology of science (including the humanities). Both philosophy and mathematics run the risk of getting lost in abstraction, and it is better to have an anchor in the empirical science of the didactics of mathematics. The General Theory of Knowledge (GTOK) presented by Colignatus in 2015 has roots in the work of three figures in Holland 1926-1970, namely G. Mannoury (1867-1956) and his students P.M. van Hiele (1909-2010) and A.D. de Groot (1914-2006). The work by Van Hiele tends to be lesser known by philosophers, because Van Hiele tended to publish his work under the heading of didactics of mathematics. This article clarifies the link from Mannoury (didactics and significs / semiotics) to the epistemology and didactics with Van Hiele levels of insight (for any subject and not just mathematics). The other path from Mannoury via De Groot concerns methodology. Both paths join up into GTOK.

This article has an implication for education in mathematics. The German immigrant Hans Freudenthal (1905-1990) was a stranger to Mannoury's approach, and developed "realistic mathematics education" (RME) based upon Jenaplan influences (by his wife) and a distortion and intellectual theft of Van Hiele's work (which distortion doesn't reduce the theft). RME is wreaking havoc in mathematics education. It is important to set the record straight, not just for accuracy, but also since there is a tendency to regard the work by Van Hiele as part of RME, so that authors like mathematician Henk Visser (ca 2005) or psychologist Ben Wilbrink (website) who want to eliminate RME for good reason also tend to erroneously denounce the work by Van Hiele. Mannoury must be saved from obliteration but also Van Hiele.

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Introduction

The following will sketch the influence by Gerrit Mannoury (1867-1956) on his students Pierre Marie van Hiele (1909-2010), (epistemology and didactics) and Adriaan Dingemans de Groot (1914-2006) (epistemology and methodology). The focus will be on the *lasting* influence. We start with a General Theory of Knowledge presented in Colignatus (2015c), and trace where it developed from.

A bit of overlap with a NWO-project 2012-2017

Around New Year 2016, and searching for more information, and following up on De Groot & Visser (2003), I discovered Henk Visser's website and the NWO-project "From Criticism to Methodology: Dutch reflections on science and society 1926-1970", Muller et al. (2012). Mannoury is one of four authors who they are studying.

I am at a distance from that project-team and have an independent interest in the subject. Obviously it is very welcome that Muller et al. have also discovered the relevance of Mannoury for our understanding of epistemology, didactics of mathematics, philosophy of science and methodology. The papers by this project-team are a welcome resource indeed. Hopefully these authors will study this present paper conversely, and include it in their library and hopefully also refer to it in their summary report for others to take notice.

To facilitate this transfer of information, I will on occasion mention connections or overlap. To start with: Van Hiele would be 18 in 1927 and meet Mannoury as his teacher at the mathematics department of the University of Amsterdam. The thesis Van Hiele (1957) is at age 48 in Utrecht under supervision by Hans Freudenthal (1905-1990), with also involvement by Martinus Langeveld (1905-1989). The influence by Mannoury on Van Hiele falls well within the 1926-1970 period of the NWO-project. The NWO-team pays attention to De Groot and then Van Hiele qualifies too.

The NWO-project focuses on what Mannoury did and not on what his students did. However, discerning the lasting influence helps to better understand the origin. The NWO-project relates Mannoury to the Wiener Kreis (Vienna) and Analytical Philosophy (Britain) but this tends to diminish Mannoury to a backwater & receiver while the Van Hiele levels of insight are a fundamental epistomological result, with a role for Mannoury as sender. Though it stands to reason that Mannoury must be appreciated for what he did, and that we should be careful not to read modern findings into what he actually did.

Mannoury and significs / semiotics

I learned of the existence of Mannoury's work and Significs (semiotics) when I was still a student in Groningen. J.J.A. Mooij referred to it in a conversation around 1977, and its relevance was immediately obvious. [1] At that time I also followed a class by Else Barth (1928-2015) on argumentation theory and the semantic tableux, developed by Evert Willem Beth (1908-1964). I pursued my path on econometrics however (with a degree in Groningen 1982). Colignatus (1992) presents a "Definition & Reality methodology", see below, actually already invented in 1990 but without time to put its design in print. In 2006 I returned to these issues (with a degree of teacher of mathematics in Leiden 2008). A reference to Mannoury and Alberts (2000) is in Colignatus (2012). It is a great pleasure to follow up now more closely on this promise from 1977.

Significs (semiotics) isn't dead but quite alive. As a teacher of mathematics I looked more closely at didactics and the role of language and symbols. I discovered various crooked elements in current tradition, so that what is called "mathematics" really isn't mathematics.

[1] "Significs" as a term was coined by Lady Welby in the 1890s. For Dutch it is better to write "Signifika" as done by Mannoury. Google shows 204 million hits for "significa", because of Spanish ("it means"). "Signifika" has 0.029, "significs" 0.0417, "semiotics" 3.8 million hits.

This is reported in Dutch in Colignatus (2008) and presented as the English book "Elegance with Substance" (EWS) in Colignatus (2009, 2015). A review is by Limpens (2010) in Euclides, the journal of the Dutch teachers of mathematics. A section below will give examples on fractions and the pronunciation of numbers. The reader is invited to study these examples closely so that it is clear what is at issue, also for epistemology.

Mannoury and Van Hiele

It is already known in (Dutch) circles of philosophy (of science) that Mannoury is related to De Groot. Lesser known are both the *relation* to Van Hiele and the *importance* of his work for epistemology and hence methodology. A main objective here is to clarify these aspects. The following will remain a sketch due to limitations of time and resources. It however suffices to put this on the map. The work by Van Hiele comes into play in two ways.

(1) It is important for the didactics of mathematics. Here it needs **defence** against misunderstanding in this realm. If this work would be denounced as irrelevant for the didactics of mathematics, it would be a fair inference that it would not be relevant for epistemology either. Thus epistemologists have the obligation to check closely whether researchers in the didactics of mathematics are really doing science.
(2) It is important for epistemology and hence methodology and philosophy of science, and here it needs **introduction** to philosophers since it apparently is unknown. Van Hiele presented his work in the publication channel of didactics of mathematics. He claimed a general result for epistemology but apparently did not publish in a journal of philosophy.

Colignatus (2015b) clarifies the meaning of "Van Hiele levels of insight" for epistemology. Colignatus (2015a) looks at the relation between Van Hiele and De Groot, where both are seen as students of Mannoury and Otto Selz (1881-1943). Colignatus (2015c) presents a General Theory of Knowledge (GTOK), based upon (a) "Definition & Reality methodology", (b) Forum Theory by De Groot (1982) and (c) Van Hiele's levels of insight.

Van Ulsen (2001:1) remarks on Beth that we are dealing with a difficult boundary of disciplines. A sketch is both incomplete and has the risk of ambiguity. I may agree 99% with everyone, but the 1% difference itself differs for each, and this distribution is not discrete (with e.g. 99% / 1% = 99 differences of view) but continuous (with perhaps more differences than people). It is not possible for me to anticipate all potential differences of views. Some readers might be tempted to take their 1% of difference as sufficient reason to reject all. Instead, readers are encouraged to hunt for what causes their rejection, and question such causes. The mentioned 1% is only an average, of course.

Structure of this paper

In the following, I will first identify the key role of didactics of mathematics, and give some examples how significs / semiotics contributes to better didactics. Subsequently I mention both the "Definition & Reality methodology" and a General Theory of Knowledge. Finally, we trace the path from Mannoury. It is best to start with a short historical positioning.

Dutch readers are alerted to that the Dutch association of teachers of mathematics NVVW in 2015 celebrated its 90th anniversary and has put the issues of its journal Euclides online. Dutch readers have now easy access to Mannoury (1930). This latter paper doesn't have the pace that a modern reader has gotten used to, but it undoubtely contains his major ideas on an introduction to significs / semiotics. **Appendix A** below has an English summary.

Appendix B contains a copy of the NWO webpage for Muller et al. (2012), just in case for when that page would disappear at some time.

Historical locus

Students of the Hebrew Bible have an age-old distinction, captured in the anagram "Pardes", between plain meaning, metaphore, referral, and secret meaning. Yahweh may speak to us in the writings by prophets, but what they write needs exegesis.

Semiotics and information theory

Semiotics is already discussed by Locke (1632-1704) and may have earlier roots in Spinoza (1632-1677). Kant (1724-1804) and Peirce (1839-1914) must be mentioned. [2]

Shortly before 1900 it was an eye-opener that the verb "to be" could be analysed in (at least) the distinction of element, subset, identity. Georg Cantor (1845-1918) developed this as set theory in mathematics, but such core notions shape our thinking, and they are crucial for didactics. In philosophy we see the development into Analytical Philosophy (IEP (2016)).

With such predecessors, Gerrit Mannoury (1867-1956) cannot really be called a pioneer. He developed the analysis and had an influence that is still important today. Beth (1956/1958:432) states about Mannoury's inaugural lecture of 1903: "He defends the view, now familiar but at that time highly provocative, that philosophy originates from linguistic problems; (...)". [3] Wittgenstein (1889-1951) had the phases of the *Tractatus* and the *Investigations*. Before the latter, Mannoury already defended the view that "the meaning of terms is their use", see Heyting (1966).

Lady Welby was in correspondence with Peirce, and one wonders whether her term "significs" and his "semiotic" denote the same, a "theory of the use of signs". For convenience I regard these as the same indeed.

It is necessary to link up to Claude Shannon (1916-2001) and information theory. A message is sent from source to destination. The message is encoded to fit the medium or channel. The intention by the sender may differ from the interpretation by the receiver. When you are interested in improving communication or didactics then it can help to analyse the various steps in this set-up.

Van Hiele and De Groot

Pierre M. van Hiele (1909-2010) and Adriaan D. de Groot (1914-2006) had classes by Mannoury. PM. It is not likely that they had classes by E.W. Beth (1908-1964). They refer to Beth however.

De Groot got his baccalaureat in mathematics and switched to psychology. He is well-known for his study of cognition in chess, which contributed to the advance of cognitive psychology. De Groot (1961) is a classic on methodology, in which he coined the term "empirical cycle". This book arrives at the negative insight that we will never know the truth. Kuhn (1962, 1970) described the social dynamics of scientific revolutions. De Groot (1982) and De Groot & Visser (2003) present Forum Theory, with the positive insight how this social dynamics could be regulated so that we might not know truth but at least have some warrants w.r.t. what we tend to accept a truth. De Groot worked on testing of students, and helped create the CITO testing bureau. It is still an issue that cognitive psychologists and psychometricians may have insufficient insight in the didactics of mathematics, so that their tests can be invalid, see Colignatus (2016c). Significs / semiotics is instrumental in arriving at valid tests (see the section below on fractions and the pronunciation of numbers).

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[2] J.F.A.K. van Benthem criticised around 1980 that I referred to Kant without having read him, but he did not explain why I for my purposes could not rely on what H.J. Störig had written, or what correction he wanted to send to Störig, so I still wonder why Van Benthem didn't deal with the subject of our conversation and went off on this kind of "criticism". See Colignatus (1981, 2007, 2011).

[3] I don't have free access to Jstor and use only the first page of this article.

Van Hiele graduated as mathematician and followed his early preference to become a teacher of mathematics. His 1957 Ph.D. thesis started out on didactics of geometry, but he in fact developed a general theory of epistemology of levels of insight, that he only demonstrated by the didactics of geometry (and "demonstration" is a term used in didactics (of geometry)). In his thesis, Van Hiele clearly states that the levels of insight are relevant for rather any subject. Van Hiele's theory is of fundamental importance.

A discussion of the relation between A.D. de Groot and P.M. van Hiele is in Colignatus (2015a). This relation is virtually non-existent. It appears that De Groot (1961, 1994) "Methodology" refers to Van Hiele (1957) but gives it a wrong presentation. Apparently De Groot was too busy writing his book to take the time to digest Van Hiele's crucial contibution to general epistemology and cognitive psychology. Remarkably, both authors had written about the didactics of geometry, namely the first presentation of (formal) geometry to students of age 12, but the communication failed. Historians of science hopefully can check whether Van Hiele and De Groot later found an opportunity to correct the issue. See Visser (2015) for De Groot on didactics of mathematics, in which Van Hiele is mentioned, but not bridging the gap. Visser (ca 2005) shows some misconceptions about Van Hiele himself too, see below.

My advice in 2008 was the creation of a Simon Stevin Institute for research in the didactics of mathematics and subsequent application in education.

Dismay on a dirty math war in Holland

There are some points that I must mention with some dismay.

To my dismay, the "review" of my book EWS by Limpens (2010) raised the question whether I would be a "wierdo", also referring to Don Quixote. [4] The editor of Euclides later admitted that she should not have allowed that "ad hominem", but she wrote this only to me in an email and not the journal itself. The proper response would have been to ask Limpens to try again, and publish the revised review. This would allow everyone to refer to a decent text, without the need to protest against the "ad hominem". One better spills some paper than obstructing science. Readers of this present article might be annoyed by the present need for a protest, but the reason lies with Limpens and the editors of Euclides. The subsequent question is whether Limpens, who works at CITO founded by A.D. de Groot, has properly informed his colleagues at CITO that their tests are invalid, at least w.r.t. fractions, like explained above. See also my letter to Parliament, see Colignatus (2015z5).

It is an empirical observation that there is a dirty math war in Holland. The main dispute is between on the one hand "traditional mathematics education" (TME) with e.g. spokesman Jan van de Craats and on the other hand "realistic mathematics education" (RME) with e.g. members of the "Freudenthal Instituut". My contribution is general science and apparently doesn't fit the perceived frameworks. At first there was appreciation of my work shown on both sides, see emails by Jan van de Craats 2008 and Jan van Maanen 2009 in Colignatus (2016b). After this, however, there was silence and they didn't refer.

A second point of dismay is as that I discovered in 2014 that Hans Freudenthal (1905-1990) had abused the work by Van Hiele. Van Hiele wrote his thesis under supervision of Hans Freudenthal (1905-1990), a mathematician and topologist who had been selected by L.E.J. Brouwer (1881-1966) to come from Germany to Amsterdam. When Freudenthal found his ability for mathematics declining, he allowed himself the choice of diverting into either the history or the didactics of mathematics. Apparently, a professor around 1965 had this freedom to think that he was qualified for such other specialisations. It would have been better when Freudenthal had supported Van Hiele as professor in didactics of mathematics. Now there is the sad conclusion that Freudenthal stole ideas from Van Hiele and sabotaged his (international) recognition, see Colignatus (2014, 2015). The International Mathematical

[4] Dutch "zonderling". Translating this as "eccentric" doesn't capture the Dutch meaning, for English academia do have some appreciation of eccentrics. But I am no eccentric either.

Union (IMU / ICMI) has a Freudenthal Award for research on didactics of mathematics, but this appears to be a disgrace.

Some points to observe:

- Freudenthal took ideas from Van Hiele and presented those as his own. This is plagiarism. Often people think that plagiarism concerns only the direct copying of texts. Hence it is better to speak about "fraud".
- The Freudenthal fraud is not crucial for the distinction between RME, TME, CME and NME. It is only a later discovery that there is this fraud. There is some human tendency to focus on such issues as fraud, e.g. to reject the whole of a paper that refers to it, but it is important to distinguish the issues.
- It is a possibility that Freudenthal's maltreatment of Van Hiele's work supported De Groot in thinking that Van Hiele's work was not important enough to read it closely. Another possibility is that De Groot didn't had little patience for Freudenthal who opposed statistical testing.

Also the role of mathematician Jan van de Craats and psychologist Ben Wilbrink must be mentioned. Wilbrink resorted Van Hiele under the umbrella of RME, instead of noticing that Van Hiele must be saved from Freudenthal's abuse. Van de Craats refers to Wilbrink and does not correct him. Van de Craats has been trained as a mathematician and not in didactics of mathematics. Trying to understand what caused Wilbrink's misconception, put me on the track to methodology and De Groot. Thus the episode was somewhat helpful in this respect. See Colignatus (2015k) and (2016a).

Finally, we might wonder whether such a "math war" has also been relevant for Mannoury and Beth. Beth was looked upon by mathematicians and philosophers. [5] For example, it is only proper when a mathematician states that something isn't mathematics when it isn't, yet, communication is difficult, and there may also be misrepresentation and bullying. Forum Theory by De Groot & Visser (2003) isn't implemented systematically, and the world still has a surprisingly long way to go.

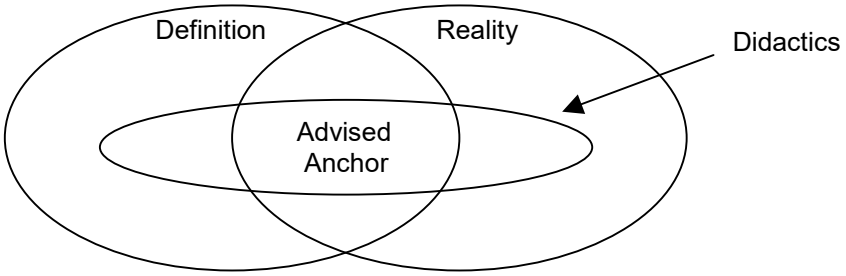
Didactics of mathematics as the anchor for philosophy and mathematics

Mathematicians are trained for abstraction while science and education are empirical by definition. Mathematicians who teach meet with real life students, will be subjected to cognitive dissonance, and will resort to their own training, which however is not targetted at empirically warranted didactics. The exercise will bedisappointing for most involved. The result can be a Western culture that mathematics is only for the gifted, which puts society on a suboptimal path. The same line of reasoning holds for philosophy. A second point is: both philosophy and mathematics are at risk of getting lost into abstraction, namely by accepting definitions that have no tracking in reality. Hence:

> The advice is that both philosophers and mathematicians take didactics of mathematics as their anchor in empirical reality, so that they are less likely to get lost into abstraction. See **Figure 1,** and see below for a discussion of the "definition & reality methodology". When you do philosophy or mathematics, you are advised to first take a degree in didactics of mathematics with practice and empirical research before pursueing your original research in philosophy or mathematics.

Figure 1. Advised core of didactics in the overlap of definion and reality

[5] Van Ulsen (2001:1): "Beth opereerde op een moeilijk grensvlak van disciplines. Hij werd door de filosofen als een wiskundige en logicus afgeschreven, door de wiskundigen en logici, evenmin positief bedoeld, voor een filosoof gehouden. Dit had indertijd, en ook nu nog, te maken met de onbekendheid van zijn vakgebied en de miskenning van het belang van zijn verdere bezigheden."



The difference between mathematics and "mathematics"

The following shows how thinking along the lines of Significs helps to create better didactics of mathematics, and contributes to clearer thinking and communication.

This section currently is in Dutch, copied from Colignatus (2016c). See that paper for the references in this section.

In de loop van 5000 jaar heeft die onderwijstraditie nogal wat elementen verzameld die niet gebaseerd zijn op aandacht voor wat daadwerkelijk gebeurt. Dit kunnen kromme en anti-didactische elementen zijn. Wiskundigen kunnen dan zeggen dat een kind of leerling onvoldoende inzicht in wis- of rekenkunde heeft, maar in feite heeft de leerling moeite om de krommigheid te verwerken. We kunnen aldus onderscheid maken tussen wiskunde en zgn. "wiskunde".

Om een veelheid aan aanhalingstekens te vermijden schrijven we nog "wiskundigen" waar ""wiskundig"-en" zuiver zou zijn. T.a.v. "realistische" en "traditionele" wiskunde zouden we ook "didactiek van "wiskunde"" moeten schrijven, want het is noch didactiek noch wiskunde.

Mijn boek "Elegance with Substance" (EWS) (2009, 2015) bespreekt het fundamentele probleem. EWS komt tot de conclusie dat het huidige wiskunde-onderwijs een herontwerp (re-engineering) behoeft. Er is een Simon Stevin Instituut (SSI) nodig om dit proces mogelijk te maken en voor de toekomst te monitoren (met ontwerp van curricula, opleidingen van leraren, onderzoek in de klas, enzovoorts), zie Colignatus (2008b) met redactie in (2012c). Zie Colignatus (2010a) voor verschillende reacties t.a.v. EWS. Zie Colignatus (2015ℓ) voor een schema t.a.v. het huidige beleidsvacuum waarin ideologieën zoals "realistisch" en "traditioneel" rekenen welig kunnen tieren. Om tot een SSI te komen is een parlementair onderzoek wenselijk, zie de petitie Colignatus (2012a).

Example: fractions

Rekenen is sinds de ontwikkeling van het decimale stelsel met Simon Stevin (1548-1620) helemaal uit-ontwikkeld. Met een beperkt aantal bewerkingen kun je alles, zie Van der Plas (2008, 2009), die echter helaas het machtsverheffen niet meeneemt. Er is nog enige discussie over haakjes en voorrangregels (Meneer van Dalen wacht op antwoord) omdat rekenmachines goedkoop kunnen zijn uitgevoerd. Maar er is geen discussie over de inhoud. Laten we dat "klassiek" noemen.

Een probleem is nog wel de traditionele notatie van 2½. Dit kan als 2 maal ½ gelezen worden (zoals 2√2). In handschrift is de spatie variabel, zeker bij leerlingen, en snel ontstaat een uitkomst als 2 ½ = 1. Klassiek is het beter om 2 + ½ zo te laten staan, het is immers ook "twee-en-een-half". Jan van de Craats (SGR) gebruikt liever 5 / 2 wat weinig didactisch is t.a.v. de locatie op de getallenlijn. Een aardige publieke discussie is Colignatus (2012b).

Onlangs trof ik een wiskundeleraar die leerlingen die van 2½ tot 2 ½ tot 1 komen "sloddervossen" noemde. Zo'n fout zal inderdaad eerder door sloddervossen gemaakt worden. Echter, deze wiskundeleraar toonde zelf een intellectueel deficiet, namelijk dat hij de stap oversloeg om eerst te erkennen dat 2½ inderdaad de conventie van vermenigvuldigen

zoals 2*x* volgt. Zou hij deze eerlijkheid hebben opgebracht, dan zou hij eerder hebben kunnen inzien dat zo'n leerling een logica volgt, en dat het de fout van de "wiskunde" is om hier een curieuze uitzondering te maken, die in een grijs verleden is ontstaan, toen wiskundigen zelf nog met de materie worstelden, en ook niet over didactiek nadachten. Het is erg verleidelijk voor wiskundigen om intolerant te zijn, want in hun denkwereld hebben zij alles met axiomatiek tot zekerheid dichtgespijkerd. Het onderwijs in wiskunde zucht onder het juk van zulke taliban die menen precies te weten hoe het moet alleen maar omdat ze nooit geleerd hebben om ook eens uit het raam te kijken.

Pierre van Hiele (1973) keek naar de mogelijkheid breuken af te schaffen door meteen met machten te werken, waarbij $1 / 2 = 2^{-1}$. Nieuw uit 2014 is een neoklassieke aanpak met de notatie $x^H = 1 / x$, uit te spreken als "per *x*", met $H = -1$ uit te spreken als "eta". Bij een notatie als 2^{-1} kunnen leerlingen denken dat ze nog iets met die -1 moeten uitrekenen. Met gebruik van *H* verdwijnt dat, en wordt een brug naar algebra geslagen. Zie Colignatus (2015a). Merk op dat de uitspraak "per *x*" ook het probleem oplost dat rangwoorden ("derde", "vierde", ...) worden misbruikt om deze breuken uit te spreken (met verwarring bij kinderen t.a.v. de vraag wat rangwoorden nu met taartpunten te maken hebben).

Wanneer een didactische krommigheid wordt hersteld dan noemen we dit "klassiek". Wanneer de didactiek vervolgens verbeterd wordt dan noemen we dit "neoklassiek". Tabel 1 geeft de onderscheidingen weer.

Tabel 1: Wiskunde versus "wiskunde" met voorbeeld 2 + ½

	Wiskunde (empirici, ingenieurs)	*"Wiskunde" (wiskundigen)*
Nieuw	Neoklassiek: $2 + 2^H$	"21st century skills" (vooral oude wijn, aansluiten bij "realistische" wiskunde)
Oud	Klassiek: 2 + ½	2½ "Traditioneel": oefenen op 2½ / 3⅓ "Realistische" wiskunde: verdoezelen

Verandering van 2½ naar 2 + ½ of beter 2 + 2^*H* is relatief gemakkelijk te bereiken, met afspraken bij het College voor Toetsen en Examens (CvTE) en aanpassing van leerboeken en CITO-toetsen, met geleidelijke aanpassing in de samenleving t.a.v. bijvoorbeeld juridische aanduidingen en medische doseringen. Het volgende voorbeeld vereist iets meer planning en inzet.

Example: pronunciation of numbers

Kinderen horen en spreken voordat zij lezen en schrijven. Het getalbegrip wordt ontwikkeld in taal en pas later uitgewerkt in schrift. Het westen schrijft van links naar rechts en de getallen komen uit het deel van India en Arabië waarin men van rechts naar links schrijft. We zeggen "negentien" en schrijven "19". We zeggen voor 21 "een·en·twintig" maar het Engels doet het beter met "twenty·one", ook al schreef Jane Austen in "Pride and Prejudice" voor 23 nog "three·and·twenty".

Het blijkt logisch om een getal uit te spreken met het grootste gewicht eerst. In 1950 stelde Delfts hoogleraar Fred Schuh voor om voor 21 "twintig·een" te gaan zeggen. De minister van onderwijs wees dat af. Noorwegen maakte in 1950 wel de overstap naar de juiste (Engelse) volgorde. Begin 2016 ontdekte ik Zijlmans (2004) waarin Duitse professor Gerritzen een vereniging heeft gevormd om voor 21 voortaan "zwanzigeins" te zeggen. Het bericht citeert Van den Heuvel – Panhuizen (FI) die gelooft dat dit niet zo nodig is. Deze wonderlijke afwijzing is voor Nederland andermaal een gemiste kans.

In het Engels zou "two·ten·one" logischer zijn, want dit maakt optimaal gebruik van het positionele systeem van getallen. Een som als 2 ×10 + 4 kun je oplossen door het uit te spreken met onderdrukking van "maal" en "plus" ("two·ten·four"). Voor het Nederlands is een systematische uitspraak met "tien" verwarrend, want 90 zou uitgesproken worden als "negen·tien". Een ontdekking in 2012 was dat het mogelijk is om in het Nederlands "tig" als tien te gebruiken. De uitspraak van 19 wordt dan "tig·negen" en de uitspraak van 90 is "negen·tig". Duitse professor Gerritzen had blijkbaar nog niet ontdekt dat het Duits van 21 ook "zwei·zig·ein" kan gebruiken. Tabel 2 geeft de categorieën.

Tabel 2: Uitspraak getallen

	Wiskunde (empirici, ingenieurs)	*"Wiskunde" (wiskundigen)*
Nieuw	Neoklassiek: 21 = twee·tig·een	"21st century skills" (vooral oude wijn, aansluiten bij "realistische" wiskunde)
Oud	Klassiek: 21 = twintig·een	"Traditionele" en "Realistische wiskunde": 21 = een·en·twintig

Verandering van de uitspraak van getallen is een intensiever proces. Het is geen spellingshervorming maar een kwestie van meertaligheid, met "Algemeen Beschaafd Wiskunde" (ABW) naast dialecten zoals ABN. Nederland zou moeten overwegen om en masse over te gaan op het tellen en rekenen met "tig" in plaats van "tien", opdat kinderen eindelijk de volle vruchten van het positionele systeem kunnen plukken. Dit kan natuurlijk ook geleidelijkaan met kinderen die in leeftijd vier op school instromen, zodat de maatschappij in tig·vier jaar kan wennen. Tijdens de lessen tellen en rekenen wordt elf uitgesproken als "tig·een". Daarbuiten mag je natuurlijk nog steeds spreken over het nationale elftal of de raad van elf (en wat is tegenwoordig het verschil). Zie meer in Colignatus (2015a), dat ook genoemd is in Colignatus (2016c:44).

Definition & Reality Methodology and a General Theory of Knowledge

Van Hiele (1957) presented a theory of levels of insight. These levels have a logical structure. The base level is in the realm of sensations. This student doesn't have a well-developed memory and relies on direct experience to feed the mind. Language is used to label properties. The second level concerns a growing awareness of relations between those properties. Obviously, this level cannot be attained when the student has no idea about or memory of such properties at the base level. The highest level is deductive, when the student arrives at a logical model.

In educational practice, students at different levels have different languages, in which the same words have different meanings. Teaching can get slow at level transitions, and didactics must be developed to facilitate such transitions.

Colignatus (1992) (2000, 2005, 2011) (pdf online) presents a "Definition & Reality Methodology". The idea is to choose definitions to best capture empirical information, so that we aspire truth by definition. The empirical problem shifts to the question whether the definitions apply or not. Popper's theory of falsifiability is less relevant when the empirical issue is applicability. (This potentially shifts the field from science to engineering.) Colignatus (1994) refers to De Groot (1982) and Forum Theory in the analysis that the Trias Politica model of democracy is deficient and requires a constitutional Economic Supreme Court to safeguard the role of economic science in economic policy making. A short memo in the RES Newsletter relating this to the 2007+ economic crisis is Colignatus (2014).

Colignatus (2015b) explains the relation of Van Hiele levels to the D&R methodology. The power of the Van Hiele levels is they can be defined logically, whence we arrive at truth by definition. It is a matter of research and engineering to identify and manage the actual processes.

Colignatus (2015c) presents a General Theory of Knowledge, that integrates these three elements: the D&R methodology, Van Hiele levels and De Groot's Forum Theory**.**

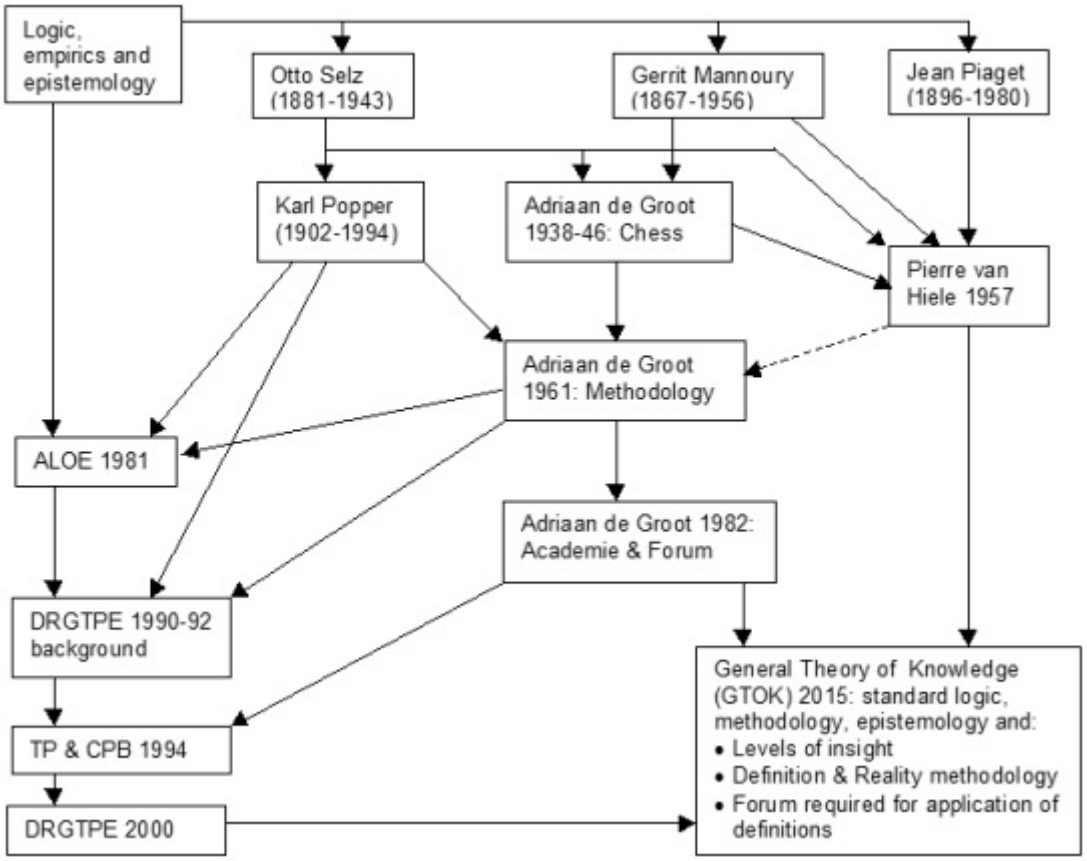
Figure 2 has been taken from there. See the discussion there. Obviously Mannoury affects his two students Van Hiele and De Groot. A deserved place is for Otto Selz (1881-1943) who influenced Popper, De Groot and Van Hiele. Recently I saw that Selz also influenced Mannoury (lost the reference).

The link between Piaget and Van Hiele is that Piaget already hypothesized levels of understanding, but linked those to age, which makes this an issue of antropology. Van Hiele started there, but eliminated age as a factor, and found the general logical relations that create a theory of epistemology.

Beth is not in this diagram. Van Hiele was in contact with Piaget and Beth was in contact with Piaget, but I am not aware of cross references. Van Hiele (1957) refers to Beth (Euclides XVI no 1 and Beth 1955) but I have not checked its importance.

Beth & Piaget (1966) do not refer to Van Hiele. They quote Mannoury but don't fully agree: "But all that mathematics is still tricked out in, its absolute character and perfect accuracy, its generality and autonomy, in a word, its truth and eternity, all this (if I may be forgiven the expression) *all this is pure superstition*!"

Figure 2. Genesis of a General Theory of Knowledge



Tracing the roots

We now have findings about Van Hiele and De Groot, which is the major part of our objective: to know what we are speaking about, i.e. (part of) the lasting influence of Significs. Tracing the roots to Mannoury now is a matter of diligence.

A basic reference is Van Ulsen (2014) and for De Groot there is Visser (2015). The key challenge is to show the link from Mannoury to Van Hiele. There are other sources, but my path was along Visser (ca 2005).

Visser (ca 2005) on Mannoury and Van Hiele

Visser (ca 2005) reviews the discussion between "realistic mathematics education" (RME) ("reform mathematics") and "Traditional mathematics education" (TME).

The paper is incomplete. In a personal communication, Visser states his view that Van Hiele is "done and over with" (Dutch: "gepasseerd station"). This cannot be found in the paper, but it is important to know, since it indicates that Visser apparently will not follow up on his study of Van Hiele. Whatever Visser states about Mannoury, Beth, Van Hiele and De Groot is shaded by this blind spot. Visser has been born in 1939 and now is 75 years of age, and we can only respect his wish to focus on his major concerns. Follow up on De Groot & Visser (2003) for example is very important too.

In my analysis, Visser (ca 2005) misreads the true situation. Mannoury and Van Hiele are quite agreed, and they are practical mathematics teachers who look at empirical reality. Van Hiele needs to be saved from the abuse by Freudenthal. It was abstract thinking Freudenthal, who misrepresented Van Hiele's analysis, stole elements, and created RME. Obviously there was an international reform movement, and Freudenthal isn't responsible for all. See again Colignatus (2014, 2015) and in Dutch the math newsletter Colignatus (2015d).

The reader will appreciate above discussion of the difference between mathematics and "mathematics". Clearly RME and TME are deficient. There is "classical mathematics education" (CME) when crooked elements are eliminated. There is "neoclassical mathematics education" (NME) when didactics are further improved (along the lines of Mannoury and Van Hiele). Clearly RME < TME < CME < NME in quality. There is positive role for Van Hiele instead of a negative one. The improvements in NME bear the stamp of the tradition of thinking along the lines of Mannoury (language, significs / semiotics) and Van Hiele (levels of insight, different use of language per level). RME by Freudenthal is only a confusion. Letting children play to discover things is useful out of respect for their personal freedom, and necessary to allow them to learn how to deal with their freedom. But when it is time to study, then we must rely on the empirical science of didactics to find the proper balance between work and play.

It is not very useful to discuss the misunderstanding on this in Visser (2005a). It is better that Visser is alerted to this issue. When he doesn't have time to study Van Hiele more closely, and to look at my discussion of NME and Freudenthal's fraud, and to correct himself, then at least he can acknowledge the existence of this other analysis, and alert others to look into it.

Still some comments are useful.

- He suggests that Van Hiele (1957) is responsible for a shift from (formal) training to "insight". This is only true to a limited extent. Van Hiele (1957) emphasizes insight, because that is his subject. He should not be quoted out of context. Whatever has been done by others (overdoing it) is not the responsibility of Van Hiele.
- Van Hiele starts with a base level of sensory perception and proceeds to the highest level of formal deduction. Visser opposes this to Mannoury (1930), who holds that thinking itself is by its nature already abstract and formal. There is no opposition here. Thought is abstract by nature (what we perceive and think isn't reality itself). The only difference is that Van Hiele is more specific about the path from sensation to formal deduction, while Mannoury can only suggest that significs must be used to find that path.
- Visser emphasizes that training and the development of skill are important for the later development of insight (like musicians practice their scales). Visser is critical of the

approach that the student is (always, fully) left to himself or herself to reinvent mathematics. He applauds that Mannoury presents a game that allows students to reinvent the rules for complex numbers – a setting that Freudenthal would label as "guided reinvention". My reading is that proper didactics is a matter of balance. Left by themselves, students will not reinvent modern mathematics. Mastering mathematics however requires the development of insight, which means a form of reinvention. None of the examples discussed by Visser (ca 2005) allow to drive a wedge between Mannoury and Van Hiele, and perhaps only seemingly so, by quoting Van Hiele out of context.

Visser (ca 2005) correctly quotes instances when Van Hiele questions drilling and favours the development of insight. This is an issue of balance. Obviously students must learn the tables of addition and multiplication, in a similar way as musicians practice their scales. However, drilling doesn't support a Van Hiele level transition. See Van Hiele (1957) (1973) for didactics that supports level transition. The game on complex numbers that Mannoury presented is welcome, when it works for a particular class. Van Hiele's objective however is not to get stuck on a single game, but to extend on Mannoury and to find a general theory.

Freudenthal's label of "guided reinvention" was inspired by Van Hiele rather than Mannoury. It seems helpful at first, but there is a downside. Once Freudenthal had developed his own terminology then he apparently thought that he no longer needed to refer to Van Hiele. Freudenthal's wife was active in Jenaplan schooling, known for independent "learning by doing", play and group activities, and this is likely to have had an influence too. Freudenthal had no training as an empirical scientics and apparently thought that his RME should work.

Personally I am very amazed that Van Hiele's work has become so abused and obliterated that even a teacher like Henk Visser with such a grasp of these aspects falls into the trap of blaming Van Hiele for RME, and trying to drive a wedge between Mannoury and Van Hiele. It is fortunate that there is the Mulller et al. (2012) project that rekindles interest in Mannoury, and provides a background for the arguments to set things straight.

Mannoury (1930)

Van Hiele (1957) refers to Mannoury's "Handboek de Analytische Signifika" (1947 & 1948). Obviously Van Hiele's list of references is composed to fit a thesis. For tracing the development I prefer to look at Mannoury (1930), i.e. his introduction in Euclides for teachers of mathematics. I assume that Mannoury used this text in his teaching to Van Hiele (age 21). Indeed, I thank Visser for calling attention to this paper, and I thank the NVVW for putting the issues of Euclides online.

Roots in Mannoury (1930) for Van Hiele (1957)

Mannoury in 1930 might have written an article on better didactics of mathematics, but apparently he wants teachers of mathematics to grow aware of the relevance of significs / semiotics in general. Those teachers might become light-bearers in general (in education).

The appendix contains a summary of Mannoury (1930), a bit more extensive to show what Van Hiele had available. (A full translation into English could be useful. Too much money is wasted on translations from English into Dutch, also for mathematics books.)

(... aspects are ...)

Mannoury (1930:45, §18) argues that didactics should aspire at a basis in signfics, upon which a later axiomatic system can be built. Van Hiele later distinguishes a base level close to the senses and a top level for axiomatics.

The different levels have different languages, in which the same words have different meanings. While a teacher may argue from the axiomatic level, the language of students would normally be quite different.

Conclusions

Appendix A: Summary of Mannoury (1930) for Van Hiele (1957)

The paper has chapters and sections (§). For each section I include some summary statements. These are labeled with the following judgements.

The label "OD" refers to what many would regard as an open door (even in 1930).
The label "OK" is decent opinion but not an OD.
The label "LR" refers to lasting relevance. This is also included in the section heading.
The combination "LR / OD" is the highest compliment.
The label "VH" refers to Van Hiele as subset of LR. We observe that all LR are also VH.
The label "VHdub" refers to a dubious influence on Van Hiele.

Mannoury (1930:1-4, §1): definition of significs

Mannoury (1930) opens with a reference to speaking. OD. This fits our attention above for the pronunciation of numbers (if necessary to explain: speaking differs from writing).

He clarifies that self-monitoring of speaking is extremely hard. OD. This supports the approach of "lesson study" with e.g. feedback from others to the teacher. Obviously, Hollywood has turned this into an industry for the field of fiction other than mathematics.

He focuses on a "speaking act" or "languate act" and calls attention to the intention of the speaker and the interpretation by the hearer. OD. The word becomes flesh.

He introduces Signifika by means of an example. A student states "the sum of a triangle is 180 degrees", which should be corrected in "the sum of the angles of a triangle is 180 degrees". The correction can be to restate the definitions. It is significs however to wonder why the student made that error, and how this origin contributes to more errors. This is thus a combination of psychology, linguistics and mathematics. Remarkably, Mannoury doesn't simply call this "didactics of mathematics". Significs as wider than this. The article only presents significs for teachers of mathematics (see its title). OD. Restatement of say Peirce. PM. Note that degrees are RME and TME, and that it is NME to take the plane itself as unit of measurement. Addition of angles is easier to grasp when angles are presented as radial fractions of the the plane.

Mannoury (1930:4-6, §2): challenge vs talent (Item-Response theory) mediated by choice of words and learning to learn (LR)

Quote: "There is perhaps no other field of study, in which the choice of words costs as much headache as in mathematics." **LR**. This notion is indeed crucial for NME. VH. Words have different meanings depending upon the levels of insight.

He also points to the individual understanding of language. OD. Individual variation is well known. The argument however is not that N = 1.

The teacher cannot pour knowledge into the student's head, while lack of knowledge can cause miscommunication. OD.

When the student has insight in significs then he or she will understand that non-comprehension doesn't necessarily imply a personal deficiency, and the student can see reason to try to bridge the gap. LR Nowadays "learning to learn", but with a wider range of tools and a better understanding of human communication.

Mathematics is man made, and there is no reason to be intimidated. LR I wonder whether another author stated this before, and it would be great when it could be labeled OD.

People can get tired. Students too. It may be required to return to a previous lesson. OD.

When even utmost effort doesn't generate the required insight, then there can still be hope. Look deeper and deeper for fundamental misconceptions, or mental blockages (psychological suppression). Call professional help (other than the math teacher). OD.

We must accept that a single person might not be able to understand even elementary mathematics (as the Pythagorean Theorem). OD. Except that Mannoury considers this a small group while it is more common to think that this will be a large group. Dutch highschool has the distinction of C < A < B < D in mathematical level, and about 15% gets D.

Talent for mathematics cannot simply be judged by grades for exams. OD. It is a challenge for A.D. de Groot to develop tests that actually can. Van Hiele developed the notion of insight, that is crucial for such talent.

Mannoury (1930:6-8, §3): Conscious, subconscious, non-conscious

Language acts have a range of conscious, non-conscious, and intermediary subconscious aspects. The degree of expertise makes how far one can get. OD.

Mannoury (1930:8-11, §4): subconsciousness

Subconsciousness is an aspect of psychology that must also be understood by teachers of mathematics. A key example is waking up (from sleep). Who dreams is not conscious but subconscious. OD. Freud's Traumdeutung 1899 was already famous.

Mannoury (1930:11-15, §5): a language act is normally not isolated

A language act with sender and receiver is normally not isolated but there is interaction. Messages go both ways. There is memory. There are subconscious expectations. There are observations also from the environment. OD.

The intention by the speaker is the cause and the interpretation by the hearer is the effect. OD. There would be more causes and effects. A speaker might say what the hearer wants to hear. But teleology is excluded.

The term "superconscious" might refer to the common meaning that is understood by sender and receiver (or in general). Significs however allows for differences. OD.

Concepts are relevant for language acts even though words need not be spoken. A dog owner may command the dog "Go to your kennel", and then the word "dog" isn't spoken, but still the notion of a dog is important for the language act. OD.

The article started with a very complex description of human communication. Then simplified to a language act. Now complicates again. The analogy is given of creating a train (by adding wagons) and managing it (driver, track manager, logistics manager, station chief, and so on). The information required is quite much. OD.

There is also "silent speaking" and thinking, comparable to a speaking act. It remains useful to split the act into a speaking cause and internal hearing effect. (A thought might surprise you.) Terminology: "word-thoughts". OD.

Mannoury (1930:15-17, §6): mental phenomena

Memory can create flashes that are so complex that one thinks that they give a full experience. One can imagine notions without having seen them. Memory tends to work in the same way for various purposes. OD.

Reading a book while other people are talking, with one's attention roaming, shows that one can see and hear without understanding (while the true magic is the understanding). OD.

Twelve distinctions (with examples) of "realisations" (Dutch "gewaarwording"): Tree rubrics (observation, memory, imagination), Two kinds: image or complex. Two intensities: superconscious versus subconscious. OD. Likely the "super" can be dropped.

(PM. The term "realistic" in RME is sometimes linked to this "realisation" (Van den Heuvel – Panhuizen). If this were the true source, then the name should be "realising mathematics education". Rather, the true source is the basic Van Hiele level, in which students sense reality and are giving names to phenomena to become familiar with them. This is in opposition to Euclidean axiomatics that starts education with the highest Van Hiele level. The later story by Van den Heuvel – Panhuizen that replaced "realistic" by mental realisation is a falsification of this historical origin, and apparently was a reply to criticism that RME gets stuck in realistic contexts.)

Mannoury (1930:17-19, §7): emotions

The distinction between positive and negative emotions is too rough and too personal. A more objective measure lies in the effect on other thoughts, as shown externally. An operalisation is: Positive emotions show from the desire of reinforcement, negative emotions from efforts to reduce them. We can observe such acts while we may not know what the thoughts were. OD.

Mannoury (1930:19-21, §8): goals and intentions, significs is a science

Two agents can have the same cognitive base but different goals and intentions, so that a language act still creates a difference between sender and receiver. OD.

Significs is not a contemplative approach (phenomenology) but an empirical science, involving observation, experimenting and statistics. The discussion in this paper gives only a rough indication. OD.

Mannoury (1930:21-23, §9): complexity of mental processes

When saying that one will return to a topic, the mind might make a note to do so indeed. People may on occasion speak or think about themselves as "he" or "she". A lecturer of a scientific speech is both speaking a current sentence and preparing the next one. The mind might feel split up into both a prompter and a speaker. Normally the speaking would be conscious and the prompting subconscious. Mannoury states that this has been established in research. OD.

Mannoury (1930:23-25, §10): levels of language

There is a difference between words that we have learned to use "by ourselves" (say in childhood) and those that come to us via explicit definition by others (parents, teachers, colleagues). The basic level of language contains notions that are non-linguistic. Words referring to such notions are basic words. The second level consists of words of which the meaning has been defined in terms of those basics words. Etcetera. OK. Mannoury will have known about Russell's theory of types (levels of language) to deal with the logical paradoxes. It may be doubted whether it really is possible to relate a "basic word" like "food" to non-verbal notions only, e.g. when there can be also a direct link to other words like "hunger".

It is impossible to develop an accurate and complete model of such a command of language for a particular person. Such a command is also subject to change. Still, the notion is relevant for research in significs, e.g. for mathematics. Accepting inaccuracy and incompleteness, it still is a useful exercise to try to recover the levels in a language. This exercise would create only a model of the language, while the actual language still has its own existence. OK. This

apparently is done in artificial intelligence and a semantic web. Encyclopedia might still contain loops.

Mathematics uses a high level language. Mathematical thought and communication implicitly activate the lower levels. There remains a distinction between actual mathematical thought and a model of its levels. Thinking will use other processes than going up and down in those levels. OK.

Also the most elegant mathematical reasoning has been born from guessing and trial and error, and it is often more luck than wisdom when this later turns out not to be an error. The true study of mathematics is not an exercise in memory but in self-discipline. How to discipline oneself is the objective of the following. OK. Mathematicians know this but somehow teachers and students may still see only the rational reconstruction. (p28: Being born is not the same as having grown up.) Education of mathematics is served by a history of mathematics.

Mannoury (1930:25-28, §11): geometry and synthetic thought

Some have said that mathematicians might be divided in visual and auditive kinds (geometry versus algebra). Whatever how this is, the distinctions matter, and it often is important to be able to manage both simultaneously. OD.

Example of a mathematical question, that poses a problem for a student, since he has been trained on cases that are too specific and that distract from the new question. The students seems incompetent but it was the teacher who trained him wrong. When the teacher studies the working papers, he can identify both the quality of the student (who uses both drawings and algebraic reasoning) and the didactic error. OK.

Mannoury (1930:28-31, §12): what is mathematical thought ?

A distinction between a novice and an accomplished mathematician may be principally that the latter has more misgivings about his own ideas. "Second thoughts are best" should be the slogan of any mathematician. OK. Mathematicians remain human. They might still jump to conclusions they have been trained in, even when they say to have checked again.

Mathematicians are aware that results must satisfy some conditions before they might be accepted by colleagues. Those conditions are not fully written down, even for axiomatics. It remains a human enterprise. OK.

Research in mathematics comes with positive and negative emotions. There are even more angles of considerations with emotional charge than in other ways of thinking. A key component of mathematical talent is to manage this multiplicity of angles. OK.

What is mathematical thought ? What scheme resides in the mind of a researcher in mathematics ? We may presume logic, also as trained in daily life, but this is not sufficient. It is a long road from children's play to the Pythagorean Theorem. (Mannoury gives an example how a play for children contains a training in logic.) OK. Logicists like Frege suggested that mathematics could be reduced to logic. Brouwer saw mathematics as more fundamental. Mannoury seems to side with Brouwer.

Mannoury (1930:31-33, §13): school language, bodily understanding

Language at school differs from language in daily life. School abstracts from non-school. An element is motivation, e.g. a shop owner looking after his cash differs from the student who wants to give a good answer or at least avoid an error. For significs, the use of language and the motivation are linked. OD.

We know that seven and three is ten. We know that it cannot be different. The tables of multiplication might be the first mantra used by all pupils. OK. Above we saw that Dutch has

the new option of "tig" for "tien". For English there is the option of twenty-one versus two-ten-one. Ten-one o'clock plus two hours is one o'clock.

Do we have something comparable in geometry ? This question is moved to section 15. OK.

Mannoury now mentions motion as different from a language act. Human experience of natural phenomena comes with a (bodily) understanding, that would differ from language. When a ball approaches you too fast then you duck. Grasping events in nature means checking this on one's experience with nature. Grasping a deduction means checking this with our logical faculties. Understanding geometry relies on these two kinds of understanding. OK. As I understand Van Hiele, the bodily understanding of nature still belongs to the basic Van Hiele level. It is a good question whether body-sense (touch, motion) needs such a special place along language, and separate from the other senses. One would assume not, and see where this becomes problematic.

Mannoury (1930:33-36, §14): experience of photos or films

We can imagine a picture without motion. We cannot imagine a film without the option to stop it at any frame. OD.

A thought about a film of a dog differs from a thought about a dog. The physiology and psychology of seeing is amazingly complex. OD.

Mannoury (1930:36-38, §15): attention and talent

Attention (and its direction) is required to solve problems. How does this relate to the mental state, with its linguistic and bodily and other aspects ? The mental eye (supported by real eyes and drawings) is influenced by a linguistic question. There are emotions and reasonings. Arriving at a solution isn't mere coincidence. There is a selection of solution strategies, e.g. based upon emphasis on elements in the linguistic question. There is this "mathematical talent" (empirically observed). OD.

Mathematical talent shows itself in geometry predominantly by the ease of managing the attention for different aspects. Most people switch between spatial views and logical reasoning, must try to get them parallel, and do not experience those simultaneously. Each switch requires the repetition and reconstruction of the partial path on that aspect. Virtuosity shown by Apollonius, Pascal, Jacob Steiner lies in the perhaps even simultaneous mastering of these aspects. OK.

The untalented can achieve much with exercise and diligence, which fickle talents do not. A condition is that one is aware of one limitations. OD. Students who get a C would get the message. It is more complex to generate motivation and awareness how this might become a B. With the advent of ICT in all realms of society, it is amazing how education of mathematics has been hesitant to adopt computer algebra and adaptive testing.

Mannoury (1930:38-40, §16): Thought, symbol, interpretation (model)

Thought itself (or the proces of thinking) differs from the manipulation of words and symbols. The first causes the latter, or, the latter accompanies it. OD. This fits Brouwer.

Calculators and computers can reproduce mathematical results, but are mechanical and not the same as thinking and knowing, with command of meaning. OD.

Mathematical thought has a purpose that is wider than most scientific fields: a purpose directed at the interaction of different mental notions. The purpose is thinking itself. Or as Brouwer expresses it: to (mentally) "jump from goal to means" in general. This purpose has a social base, whence mathematics is is related to the humanities and the social nature of man. Mathematics is no, as often thought, a collection of abstract truths independent from human needs. A mathematical formula is not true or false, but only applies or not, depending upon

the interpretation. OK. Dirk Struik's history of mathematics paints the social background. Lev Vygotsky emphasized the social base of learning. Mannoury's terminology differs from distinguishing truth by definition (tautology) from empirical relevance of such a truth.

""One plus one" is thus and only thus "two", because we (Dutchmen) mean that "two" is practically the same as the word sequence "one plus one". Words that are synonymous will still have different mental associations." OK. This is a problematic quote. "Practically the same" is a qualifier that destroys mathematical identity. Obviously, an expression with "plus" would have more meaning than an expression "two" without it. We can suppose that there would be no disagreement when we distinguish expressions from interpretations, and adopt Peano Arithmetic (for the arithmetic interpretation of "plus").

This is an aspect of the difference of (understanding of) language between the novice and the accomplished mathematician. OD. Van Hiele's innovation is that there are levels of insight with different languages (same words with different meanings). This is more than only the open door that masters understand more than novices (normally).

Mannoury (1930:40-42, §17): introduction to geometry, notion of proof, natural vs mathematical language, limited value of drilling, get students to communicate amongst themselves (LR)

One of the main didactic problems is that the novice in geometry is asked to prove all kinds of things, while it is impossible for him to see that such a proof is required. That two triangles of which the corners overlap, also fully overlap. That all lines in a plane in point A are perpendicular to one particular line. That a circle is divided by a diameter in two equal parts. it is obvious or adacadabra. OK. Mannoury recalls that he himself as a school boy associated the expression "kicking in of open doors" with his class in geometry. (Van Hiele's selection of geometry for his 1957 thesis is not surprising, given the traditional education with Euclid.)

This didactic problem is caused by the different meaning of "proof" in mathematics versus natural language. Proof in natural language means "convince someone". (Use arguments to replace initial resistance by agreement.) Mathematical proof is different, and should rather be called deduction. (Dutch "afleiding" has the confusing second meaning of "distraction".) Show that a property is contained in earlier definitions. (Proof theory itself is still subject to mathematical debate.) A student who understands this conceptual difference will be less distracted by the associations in natural language. OD.

Students (of age 12 ?) who get this, need rely less on insight and can observe that much of the work is of an operational kind, and they can derive a great deal of satisfaction from it (even drawing a tree of the dependences and hanging this up at home). OK

This problem with the word "proof" is only an example.The phenomenon holds in general for mathematical terms that have been borrowed from natural language. Who is not aware of the different contexts and meanings, is at risk of confusion at almost each word, and with the consequence of failure and loss of interest. **LR / OD / VH**.

Much will be won when teachers would see that students develop only slowly in language and thought, before they become familiar with thinking in mathematical manner. This development cannot be forced by endless practice and learning of lessons (drilling). The main factor is communication (by mouth) between teacher and student, and, even more, between students themselves. They can explain issues to each other often better than teachers. **LR / VH**. This fits Van Hiele: development, last stage is mathematics proper (deduction), distrust of drilling, language depending upon level (the teacher's level may be incomprehensible for students, so let them use their own language).

Mathematics is thinking, and thinking involved dealing with language. OD. There is still the difference between thought and language.

It must be feared that teachers will tend to stick to the traditional approach: to instruct students to sit still and not tell others. OD.

Mannoury (1930:42-45, §18): Example: what is a construction ? Guided reinvention (LR)

A construction by means of pen and paper differs from a mathematical construction (in the mind). Students may see that a triangle will not rust and decay, but still be less aware that a drawing is insufficient and must be accompanied by a proof. And be even less aware that this relates to the "definition, theorem, proof" approach. Still, this awareness needs to come about, if we want to get mathematical competence. OD. Known to Thales and Euclid.

This also requires the awareness that every result can be put into a theorem (but not every theorem in the form of a result (?)).

Working with ruler and compass helps to keep the body-sense active when it is at risk of being overwhelmed by (the distractions of) logic. But it has false positives too. Drawing a line between A and B makes students forget that this essentially means only giving a name to the (mathematically already existing) AB. Application of the Euclidean postulate is a mental act. OD.

A definition gives the general situation. A construction is an application and may require a proof that it is so. A construction may also be seen as its definition in terms of the postulates. OD. We can label this an open door, even though Mannoury wonders whether all teachers of geometry are aware of this. Note that the Pythagorean Theorem requires proof in the Euclidean system, while it would also be possible to define it directly as the distance measure (with definition and no need for proof, only need for applicability).

Imagine a student (age 12 ?) with ruler and compass, with his "student language". What must be done or not done, to allow him to arrive at mathematical competence ? Don't spoil the situation by direct instruction. Who would start the Introduction with an exposition about the formal and symbolic meaning of the postulates, would be as foolish as the traditional teacher, who started the Introduction to Geometry for his students of age 11 or 12 with a philosophy on points, curves and planes. The only advantage of such an approach is that the minds of the students are not affected since everything is soon forgotten. **LR / OD / VH.** Via the development of Van Hiele levels of insight we are aware that the formal insight is the last stage. Learning is from concrete to abstract. Learning is from vague to specific. Do not confuse vagueness with generality (abstraction), see Colignatus (2015z6).

Do: get students to grow aware / allow them to arrive at the insight about the regularities, that can be observed in their speaking and thinking about physical phenomena in general and their experiments with ruler and compass in particular. How this "get them grow aware" must be done exactly, cannot be specified in general. It does not really matter how. Every method is good, and using only one method is boring. **LR / OD / VH**. Van Hiele helped identify the level transitions, that require special attention.

Do: keep one's eye on the ball. The purpose of the Introduction to Geometry is to lay the foundations using significs, upon which the axiomatic system can be built (much later). **LR / OD / VH**. Mannoury identifies the order. Van Hiele identifies the levels.

Mannoury (1930:, §19):

Mannoury (1930:, §20):

Mannoury (1930:, §21):

Mannoury (1930:, §22):

Mannoury (1930:, §23):

Mannoury (1930:, §24):

Mannoury (1930:, §25):

Mannoury (1930:, §26):

Mannoury (1930:, §27):

(... incomplete yet ...)

Appendix B. Webpage of NWO-project by Muller et al. (2012)

QUOTE February 17 2016

From Criticism to Methodology: Dutch reflections on science and society 1926-1970
Summary

The methodological and philosophical reflections caused by the dramatic changes in science in the 20th-century have not lost a whiff of their urgency. When today we feel a need for clarity and rationality of language, it is good to be aware of the fountain of concept-critical thinking created over the past century. From the beginning of the 20th-century, concerns about ambiguities in communication in science and in general have inspired a community of intellectuals in The Netherlands to reflect thoroughly on language and understanding.

Parallel communities elsewhere in Europe, notably the Vienna Circle in Austria, and analytic philosophy originating in the Cambridge of G.E. Moore and B.A.W. Russell, developed into a dominant tradition in philosophy of our days. Within the Dutch concept-critical community a major stream of thought was developed by the Signific Circle. From Criticism to Methodology sets out to study the Dutch history of analytic philosophy, in the manner of the history of ideas, and thus to include a discussion of the content of these ideas and show their relevance for present concerns.

In contrast to the continuing interest in the history of analytic philosophy in other countries, notably the Wiener Kreis and British analytic philosophy, the Dutch concept-critical tradition has been understudied.
We want to bridge this gap in the Dutch and European history of ideas, with an emphasis on the period from 1930 till 1970. From Criticism to Methodology will achieve this by studying the works of the leading persons involved: Gerrit Mannoury (Significs, or theory of means of communication, and didactics), David van Dantzig (Significs, foundations of probability theory, methodology, information theory, Unesco), Evert Willem Beth (logic, foundations of the exact

sciences, philosophy of science and methodology), and Arend Jan Heyting (intuitionistic mathematics and logic). Beyond the published works, archives in Amsterdam and Haarlem hold a wealth of unpublished manuscripts, typescripts, and letters waiting to be studied and disclosed. The results will offer a long due contribution to the international discourse on the history of analytic philosophy, logic, methodology and philosophy of science, which have been attracting intense and wide attention over the past decades.

Products

Professional publication

G. Alberts(2013): Dirk Jan Struik (1894-2000) pp. 271 - 275
G. Alberts(2013): Tussen Tannery en Klein: polariteit in de geschiedschrijving van de wiskunde in België en Nederland pp. 173 - 186

Publical information

M. Kirkels(2013): Gerrit Mannoury?s signific philosophy
P. van Ulsen(2013): Organisaties en genootschappen
P. van Ulsen(2013): Wiskunde-onderwijs
P. van Ulsen(2013): Introductie van logica en wetenschappelijke wijsbegeerte aan de Nederlandse universiteiten
G. Alberts(2013): Tussen Tannery en Klein: polariteit in de geschiedschrijving van de wiskunde in België en Nederland

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UNQUOTE

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