Dutch Teachers of Math Day, November 9 2013

https://www.nvvw.nl/15884/subthema-d-diversen

D5. The algebraic approach to the derivative

Algebra allows a different didactics of the derivative. The emphasis is on didactics. Some scope on new theory.

Thomas Cool / Thomas Colignatus
Econometrician (Groningen 1982) & teacher of mathematics (Leiden 2008)
http://thomascool.eu

- 15 minutes: introduction
- 5 minutes: questions for clarification
- 15 minutes: continuation of the presentation
- 20 minutes: discussion
- 2 minutes: conclusion

Summary from the programme booklet

Algebra allows a different didactics of the derivative. Didactics has the standard notion of the “procept”, in which a mathematical concept can also be a process, or conversely. When we apply this to division then there arises a distinction between normal division as a static concept and the new notion of dynamic division with algebraic manipulation. A corollary is the new algebraic approach to the derivative. In this approach limits are no longer needed to construct the derivative. Cauchy and Weierstrasz remain relevant for university but are no longer needed for secondary education. Students can directly understand the algebraic approach. This allows easier transfer to courses in physics and economics. See the PDF of *Conquest of the Plane* at: http://thomascool.eu/Papers/COTP/Index.html

Contents of the presentation

(1) *procept* (process – concept), Gray & Tall 1994

(2) variable: not only number but also name

(3) standard division with limits

(4) static versus dynamic division

(5) a corollary is the derivative

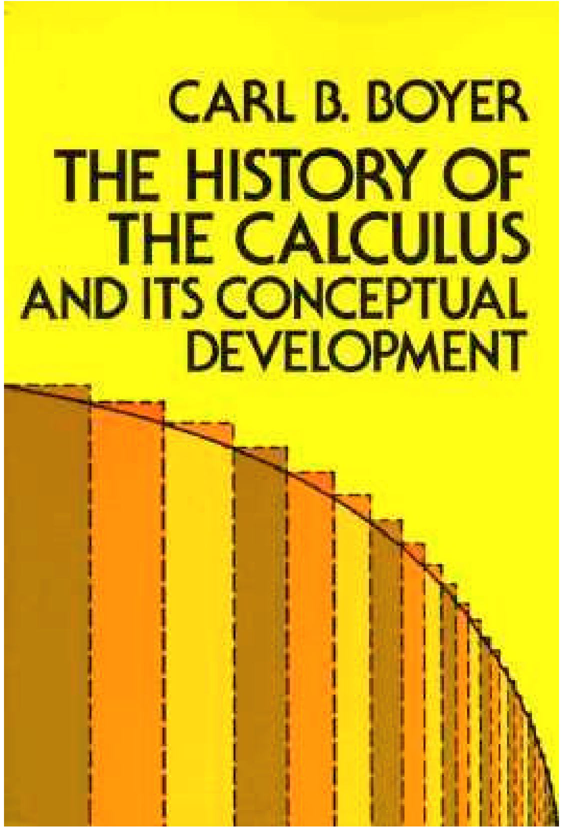
(6) rather start with surface (integral)

(7) problems in the current approach

(8) questions to start up the discussion

By the way a bit of literature

https://archive.org/details/TheHistoryOfTheCalculusAndItsConceptualDevelopment



http://www.fisme.science.uu.nl/nl/handboek/hoofdstukken/

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Handboek wiskundedidactiek

Auteurs **Paul Drijvers**
Anne van Streun
Bert Zwaneveld

Bedoeld voor aanstaande wiskundeleraren in (de tweede fase van) het voortgezet onderwijs, maar ook voor hun collega's die daar al werkzaam zijn, voor opleiders in het eerste- en tweedegraads gebied, auteurs van schoolmethoden en andere belangstellenden.

Prijs € 34,00
nieuwe druk

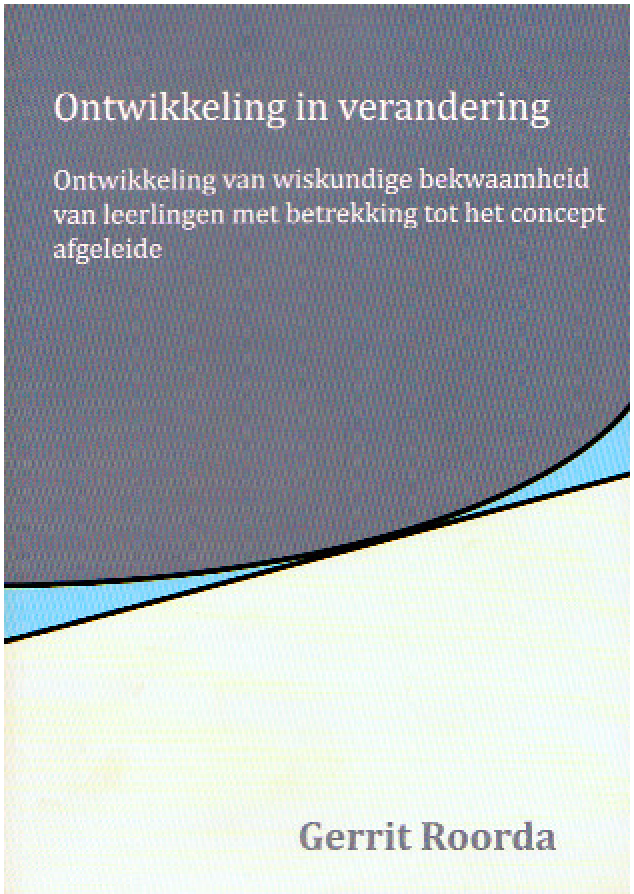
Uitgave 3e druk, 2013.
ISBN 978-90-5041-130-1
400 pagina's

Inhoud

Het Nederlandse wiskundeonderwijs ziet er heel anders uit dan in 1974, toen het invloedrijke boek Didactiek van de wiskunde van Joop van Dormolen verscheen:

Thesis by Gerrit Roorda

http://www.rug.nl/staff/g.roorda/proefschriftGerritRoorda.pdf



Website David Tall

http://homepages.warwick.ac.uk/staff/David.Tall/index.html

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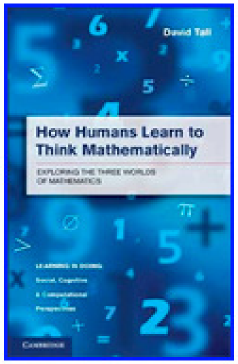
David Tall - Professor in Mathematical Thinking

Welcome to the HOME page of my website.

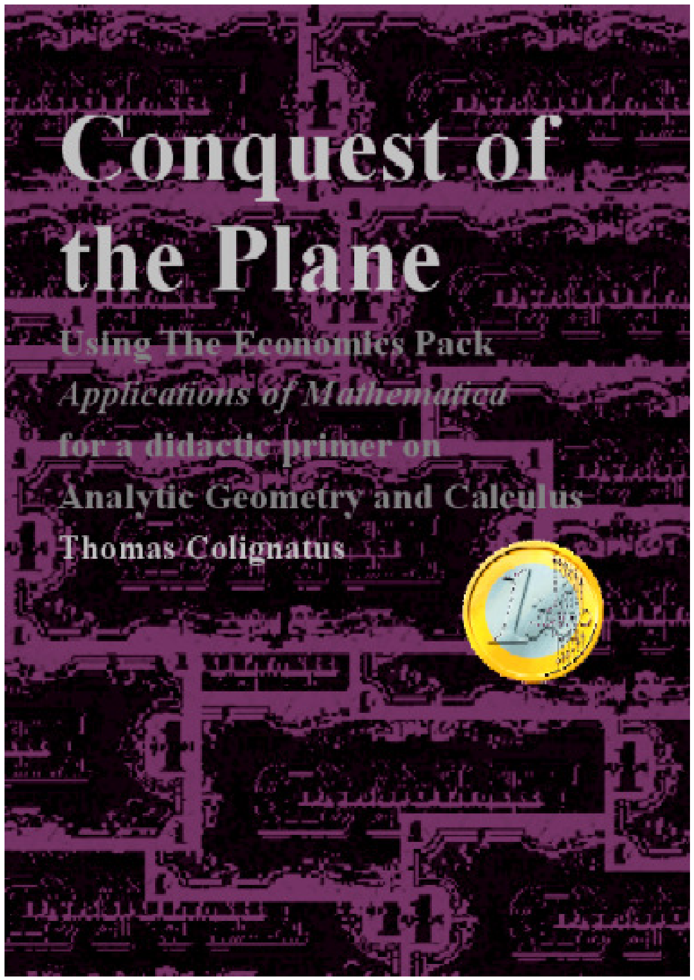
My book on ***How Humans Learn to Think Mathematically*** has been published in September 2013 (Paperback $39.99, Hardback $99.99). On Amazon.co.uk the price is Paperback £25.32, hardback £48.63 and other deals include **The Book Depository** which currently quotes world-wide delivery including postage at a competitive price.
A number of new papers (and drafts) have been added recently. Feel free to use information about my **research** as a resource, or **download** a paper. There is NEW **news** about recent changes on this site (made on **Wednesday, October 2nd 2013)** , and also **drafts** of earllier papers and **links** to other sites of interest.
See below for more information, including my students and my supervisors/mentors back via Newton and beyond.



- information on several **research themes** with links to relevant papers:
  - cognitive development | concept image | cognitive units | cognitive roots | generic organisers
  - procepts | algebra | limits, infinity & infinitesimals
  - visualization | calculus (& computers) | computers in school and college
  - problem solving | advanced mathematical thinking | proof
  - *three worlds of mathematics*
  - *lesson study*
  - *How Humans Learn to Think Mathematically* NEW

http://thomascool.eu/Papers/COTP/ConquestOfThePlane.pdf (2011)

Original presentation in “A Logic of Exceptions” (2007)



What we will *not* do here

- Abstract algebra of ‘derivation’
  http://en.wikipedia.org/wiki/Derivation_(abstract_algebra)

- Meadows & Fields (division by zero)
  http://staff.science.uva.nl/~janb/#researchprojects

Somewhat saddening videos on the internet:

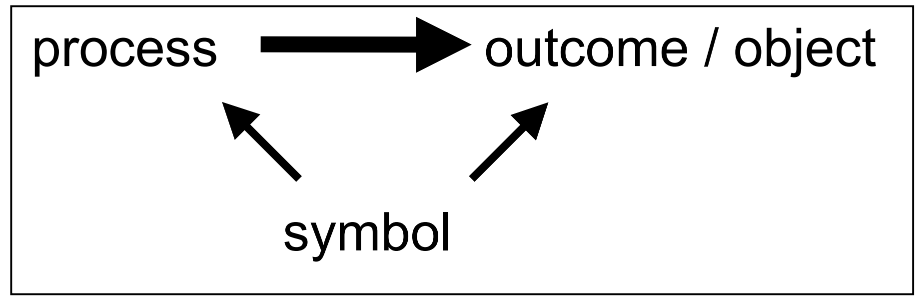
- http://www.khanacademy.org/math/calculus/limits_topic/continuity-limits/v/fancy-algebra-to-find-a-limit-and-make-a-function-continuous (zie met name minuut 4)

- http://ocw.mit.edu/courses/mathematics/18-01-single-variable-calculus-fall-2006/video-lectures/

(1) ***Procept*** (process-concept), Gray & Tall 1994

http://www.warwick.ac.uk/staff/David.Tall/themes/procepts.html

elementary procept:



An *elementary procept* is an amalgam of three components: a *process* which produces a mathematical *object* and a *symbol* which is used to represent either *process* or *object*.

A *procept* consists of a collection of elementary procepts that have the same object.

> $\sqrt{2}$: grab the calculator or manipulate it as a symbol ?
> Verb versus noun, e.g. “to ride” versus “a ride”.

Gray & Tall's notion of procept improved upon the existing literature by noting that mathematical notation is often ambiguous as to whether it refers to process or object. Examples of such notations are:

> $3 + 4$: refers to the process of adding as well as the outcome of the process.
>
> $\sum_{n=0}^{\infty}(a_n)$: refers to the process of summing an infinite sequence, and to the outcome of the process.
>
> $f(x) = 3x + 2$: refers to the process of mapping x to 3x+2 as well as the outcome of that process, the function f(x).

http://en.wikipedia.org/wiki/Procept

(2) Variable: not only number but also name

A *real variable* can be seen as a procept:

Variable:

$$\begin{tikzcd} \text{choose} \arrow[rr] & & \text{chosen value} \\ & x \arrow[ul] \arrow[ur] & \end{tikzcd}$$

("*x*" is the name)

Numerical: *x* is always regarded as 'some number'. This is the conceptual world of the derivative following Weierstrasz.

Algebra: with "x" a name (a name now with a single letter), *x* is called a "symbol", and this symbol can be manipulated according to some rules. This is our conceptual world here.

E.g. eliminate the brackets: $(x+1)(x-1)$

(3) Standard division with limits

Standard division is in the numerical conceptual world.
Standard division causes a lot of work.

$$f[x] = \frac{x^2 - 1}{x - 1} = \begin{cases} x + 1 \textit{ if } x \neq 1 \\ \textit{not defined if } x = 1 \end{cases}$$

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

f[*x*] is *continuous* in point *a* <u>*iff*</u> the limit there is *f*[*a*].

$$g[x] = \begin{cases} \frac{x^2 - 1}{x - 1} = x + 1 \textit{ if } x \neq 1 \\ 2 \textit{ if } x = 1 \end{cases}$$

Proposal to get from vague to sharper use of language.

Presently, when we say that a function $f[x]$ is *not defined* when $x = a$, then we mean:

- either that $x = a$ is in the domain of f, so that only the value $f[a]$ is not available in the range,
- or that $x = a$ is not in the domain of f,
- or some confused combination of these ?

Proposal to use sharply henceforth:

- that $x = a$ is not in the domain
- so that the domain can be extended with $x = a$
- so that the range can be extended with $f[a]$ = the value of the limit of $f[x]$ for $x \to a$

In the derivative the difference quotient $\Delta f / \Delta x$ is not defined for $\Delta x = 0$. Domain and range are extended by the differential quotient with the limit for $\Delta x \to 0$.

(4) Static versus dynamic division

Algebra as conceptual world. Explicit domain manipulation.

Static division: y / x = numerator / denominator, like above, thus not defined for $x = 0$.

Dynamic division: y // x = {y / x, unless x is symbolic: then assume that $x \neq 0$, simplify y / x, declare the result valid for $x = 0$ with extension of the domain}.

- in computer algebra: y // x = Simplify[y / x]
- 4 // 0 = 4 / 0 and remains undefined
- for variabele x we get x // x = 1 for all real x
- a denominator ($x - x$) is not a variable because $x - x = 0$

There is a manipulation of the domain:
When the domain of the denominator contains zero then this is not considered during the process of simplification, but the domain becomes relevant again for the result.

(5) A corollary is the derivative

The derivative then is defined with this program:

$f\,'[x] = \mathbf{d}f \,/\, \mathbf{d}x = \{\Delta f \,//\, \Delta x$, then choose $\Delta x = 0\}$

in which the domain of Δx is first neglected at 0, then extended with 0 and subsequently confined to 0.

Distinguish *this definition* and *didactics with it* (see below):

- The definition is not the problem (it is a definition)
- The (didactic) issue is why you would want to use the notion of ‘derivative’
- The didactics of the derivative is simpler because it doesn’t rely on the notion of a limit (anyhow in division)
- Does this approach allow for more attractive didactics ?

Compare

(a) $\lim\limits_{\Delta x \to 0} \dfrac{\Delta f}{\Delta x}$

clarity in the manipulations in algebra and domain

(b) $\underset{\Delta x = 0}{\text{set}}\ \Delta f \,//\, \Delta x$

clarity in the order of the manipulations

(c) $\{\Delta f \,//\, \Delta x$, set $\Delta x = 0\}$

Notice:

(a) The standard development of the derivative has simplification as well. E.g. the difference quotient for x^2:

$$\frac{(x+\Delta x)^2 - x^2}{\Delta x} = \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} = \ldots$$

(b) A reader agrees: “Why resort to limits, and why not simplify before you get close to 0 ?”

(c) The manipulation of the domain in *y // x* only makes explicit what already is being done implicitly.

(d) The limit for $\Delta x \to 0$ in the differentiaal quotient is equivalent to {simplify, extend the domain of the difference quotient with 0, then choose $\Delta x = 0$}.

P.M.:

The limit is not a process but static predicate logic.

For *f*[*x*] and any *x* we define *f'*[*x*], with ε > 0 and δ > 0:

$$f'(x) = d \textit{ if and only if for every } \varepsilon \textit{ there is a } \delta \textit{ such that when}$$
$$0 < |\Delta x| < \delta\,,\ \left|\frac{f(x+\Delta x)-f(x)}{\Delta x} - d\right| < \varepsilon$$

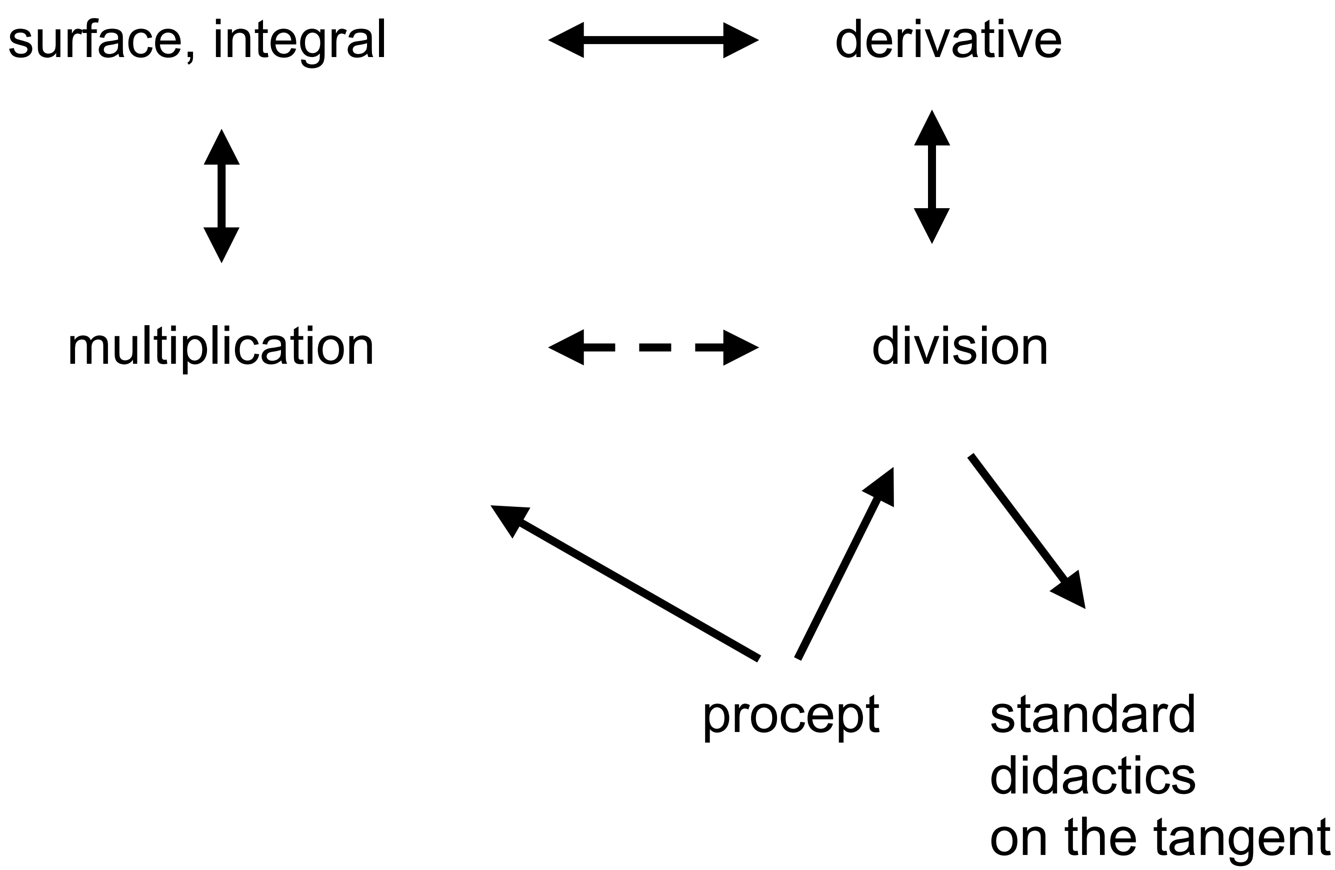
Stories like ‘it goes to zero’ are heuristical only, and create a conceptual world that is not necessarily relevant for the derivative.

It remains vague what ‘it is zero’ means.

PM. The brackets in *f*(*x*) are confusing, rather use *f*[*x*].

(6) Rather start with surface (integral)

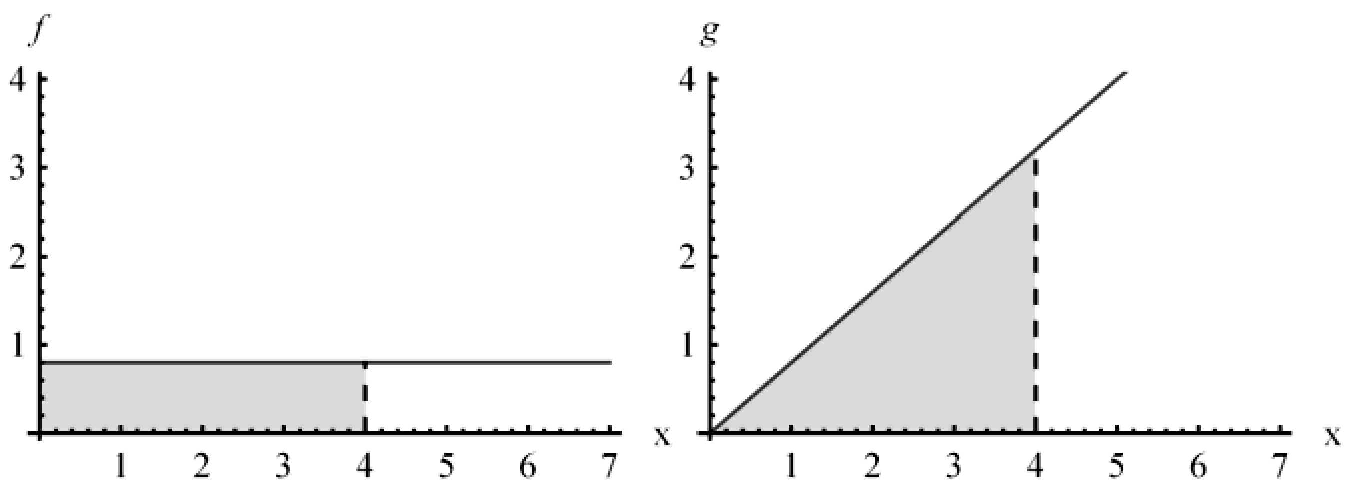
Where to start in the didactics of calculus ?



COTP starts with surface (p149):

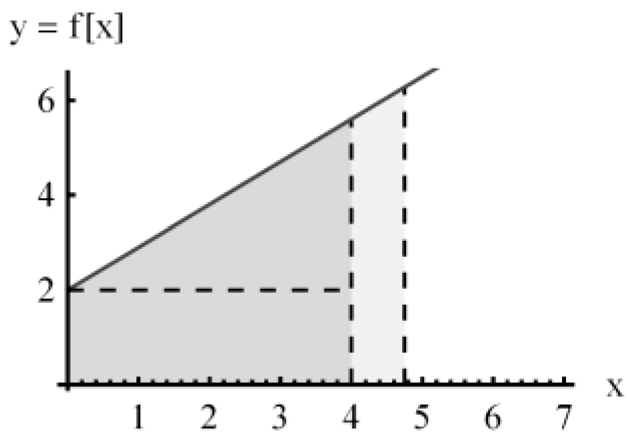
Calculus concerns the measurement of surface between a function and the horizontal axis. A key aspect is also the change in surface. The basic cases of rectangle and triangle are in the following graphs.

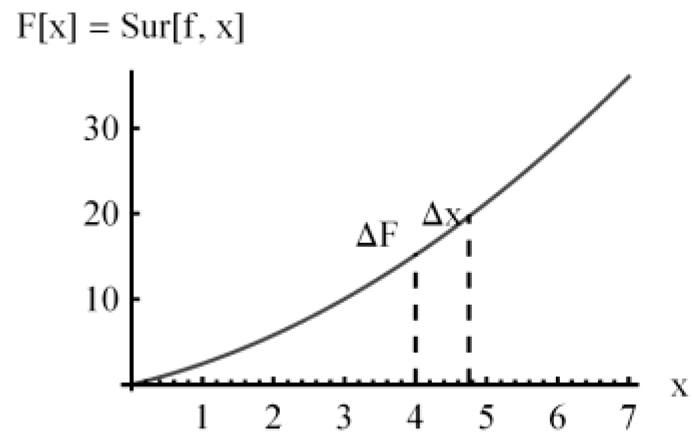
- The surfaces under $f[x] = 0.8$ and $g[x] = 0.8\,x$, for x over the interval [0, 4].



The surfaces under f and g for x over [0, 4] are easily calculated. The constant function f gives a rectangle 0.8 * 4 = 3.2. The ray function g gives a triangle ½ $h\,w$ = ½ (0.8 * 4) * 4 = 6.4. Let us make it more formal.

$F[x] = \mathrm{Sur}[f, x]$ gives the surface between $y = 0$ en $y = f[x]$, and between $x = 0$ and x itself. For example for $F[x] = x^2$:





$$\Delta F[x] = F[x+\Delta x] - F[x] \approx f[x]\ \Delta x$$

$$f[x] \approx \Delta F[x]\ //\ \Delta x$$

$$f[x] = \{\Delta F[x]\ //\ \Delta x,\ \text{zet } \Delta x = 0\} = F'[x]$$

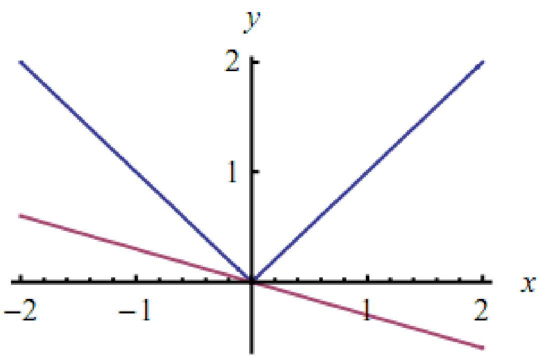
This justifies the definition of the derivative.

A function gives the rate of change of the surface under the function.
(Aside of the constant of integration.)

11.4.2 Derivative and slope of abs[x]

What is the slope of $|x|$ at the origin ? What are the tangent and tangent line ?

- Abs[x] and an example line through the origin.



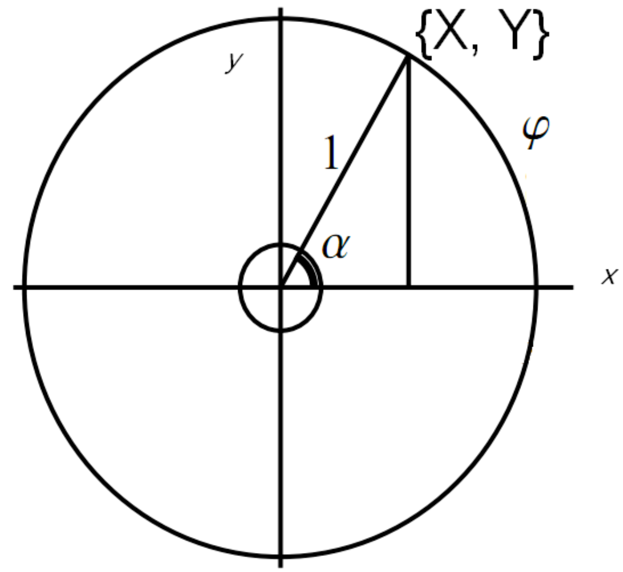
- Take Abs'[x] = Sgn[x] in general. At x = 0: ΔF // Δx = (|0 + Δx| - |0|) // Δx = |Δx| // Δx = Sgn[Δx], set Δx = 0.
- For x < 0 the tangent is –1 and for x > 0 the tangent is 1, thus in x = 0 the tangent is not defined.
- The standard approach uses the derivative Abs'[x] for the tangent, but this is not defined for Abs'[0]. We still don't need the didactics of limits to the left and to the right.

As the horizontal and vertical co-ordinates are indicated with $\{x, y\}$ it is useful to indicate the co-ordinates on the unit circle with $r = 1$ with capital $\{X, Y\}$.

The unit to measure angles: the plane *itself*.

Angle α can be found on the angular circle with $r = 1 / \Theta$ and thus circumference 1.

Arc $\varphi = \alpha\, \Theta$ is measured on the unit circle.



ur = unit radius circle

Θ = 1 archi = 2π

$X_{ur}[\alpha] = X[\varphi] = \mathrm{Cos}[\varphi]$

$Y_{ur}[\alpha] = Y[\varphi] = \mathrm{Sin}[\varphi]$

Result of *Conquest of the Plane* (COTP)(2011):

It is possible to build up the secondary school programme with the algebraic approach and to present deductions and proofs in a transparent manner:

(a) integrals and derivatives of the standard functions
- polynomials
- exponential functions and recovered exponential functions ("logarithms")
- trigonometry (xur, yur, tur) ("cos, sin, tan")

(b) as well:
- rules for integration and differentiation
- definition of *e*
- partial derivative (should also be in the programme)

P.M.: COTP has also other results.

(7) Problems in the current approach

Textbook *Getal & Ruimte* (2006), VWO B, part 1, p104: always writing “lim” ...

Differentieerregels aantonen

Het berekenen van de afgeleide heet **differentiëren**.
Met de definitie van de afgeleide zijn regels voor het differentiëren aan te tonen.

Je kunt het voorbeeld ook oefenen met de applet
De afgeleide van $f(x) = ax^2$ op de ▸cd-rom.

voorbeeld

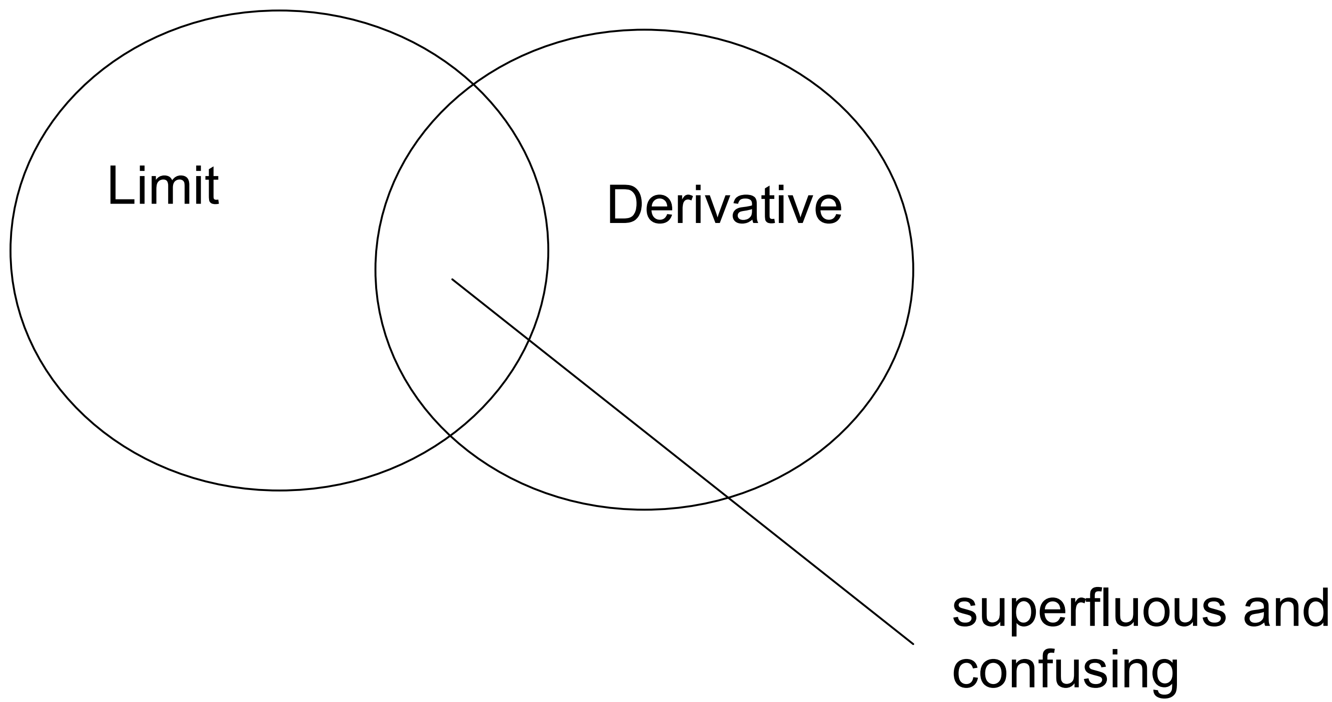
Toon aan: $f(x) = ax^2$ geeft $f'(x) = 2ax$.

Uitwerking

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{a(x+h)^2 - ax^2}{h} = \lim_{h \to 0} \frac{a(x^2 + 2xh + h^2) - ax^2}{h}$$

$$= \lim_{h \to 0} \frac{ax^2 + 2axh + ah^2 - ax^2}{h} = \lim_{h \to 0} \frac{2axh + ah^2}{h} = \lim_{h \to 0} (2ax + ah) = 2ax$$

Limits are now regarded as essential for calculus. But they can be eliminated and can be regarded as confusing. Limits remain relevant for other cases.



Mathematical competence now develops slower than needed. Roorda Ch. 3, “competence”, p27

In het concept afgeleide zijn verschillende aspecten en relaties tussen aspecten te identificeren. De wiskundige Thurston (1994) stelt dat mensen onderdelen van wiskunde op verschillende manieren begrijpen en illustreert dit met het concept afgeleide:

> *The derivative can be thought of as:*
>
> *(1) Infinitesimal: the ratio of the infinitesimal change in the value of a function to the infinitesimal change in a function.*
>
> *(2) Symbolic: the derivative of* x^n *is* $n \cdot x^{n-1}$*, the derivative of sin(x) is cos(x), the derivative of* $f \circ g$ *is* $f' \circ g * g'$, etc.
>
> *(3) Logical:* $f'(x) = d$ *if and only if for every* ε *there is a* δ *such that when* $0 < |\Delta x| < \delta$, $\left|\frac{f(x+\Delta x)-f(x)}{\Delta x} - d\right| < \varepsilon$
>
> *(4) Geometric: the derivative is the slope of a line tangent to the graph of the function, if the graph has a tangent.*
>
> *(5) Rate: the instantaneous speed of* $f(t)$*, when t is time.*
>
> *(6) Approximation: the derivative of a function is the best linear approximation to the function near a point.* ?
>
> *(7) Microscopic: the derivative of a function is the limit of what you get by looking at it under a microscope of higher and higher power.* (Thurston, 1994, p.3) Tall

Het gaat in deze opsomming volgens Thurston niet om verschillende definities

Transfer by students of knowledge from the course in mathematics to courses like physics and economics. Connecting didactics by the teachers of those courses.

This has always been a problem.

Elements in an explanation:

(a) Physics and economics don’t quite use limits

(b) The math course is slow in getting to the derivative since it is made more difficult than necessary

(c) The math course focusses on change, difference quotient, slope, direction, coefficient of change ... while it is better to start with surface and see the derivative as a ... derivative

(d) Math didactics oscillates between “Euclidean axiomatics” and “realistic math” instead of developing into an empirical science targetted on what students understand.

(8) Questions to start up the discussion

(1) What is not clear or causes immediate questions ?

(2) Is the approach already interesting for the training of teachers, to alert them to the new didactics and the pitfalls of the current approach ?

(3) Is the approach already interesting for higher education where math is used (other than math majors) ?

(4) The Dutch highschool diploma VWO-B p7, Domein Bb, point 9.3 requires: “to use the differential quotient to give a local linear approximation to a function”. When a student understand calculus via the above, it might not be too difficult to explain the differential quotient. Can we start up a process of change: acceptance in the field, trials, adapted textbooks, training of teachers, eventually reformulation of the exam requirement ? Why not ?

Addendum: From the discussion on November 9 2013

(1) In "Solve x (x + 2) = x for x" it is tempting to divide by x "because it is allowed now".

COTP p57: x is not a variable here but an unknown quantity.

Solutions are either $x = 0$ or $x \neq 0$ and in the last case we can indeed divide by x, which gives $x = -1$. Or bring all to one side: $x (x + 2) – x = x (x + 2 –1) = x (x + 1) = 0$ so that we directly see the two roots.

We may interprete "Solve for x" as "determine the domain of x for which the equation is true", so that x is a variable in the domain {-1, 0}. In that case we can uphold that x is a variable. In that case it still remains a matter of choice when to use y / x or y // x. Dynamic division rather holds for proportions than for equations like the above.

(2) In the Dutch highschool programme for the highest level of mathematics limits have actually been eliminated already. See http://www.slo.nl/organisatie/inDeMedia/2008/S45C-108070809591.pdf, Jenneke Krüger, Euclides 83-8, p375: “Though limits are in a subdomain again it is not intended to develop this into a very broad part. The limit is regarded as a necessary concept for the introduction of the derivative and for the study of asymptotic behaviour of functions.” In practice teachers speak about it vaguely and for asymptots one tries specific numbers.

For Dutch math teachers the argument in COTP, that the limit for the derivative can be eliminated, is superfluous.

Answer: (a) limits are important, and apparently removed with the wrong reasons, (b) use of dynamic division restores the lost exactness (if it existed) in transparent manner.