Memo

About: Logical errors in the standard "diagonal argument" proof of Cantor for the power set
To: Bas Edixhoven, https://www.knaw.nl/nl/leden/leden/6203
From: Thomas Colignatus, http://thomascool.eu/Papers/Math/Index.html
CC: Jan Bergstra, https://www.knaw.nl/nl/leden/leden/11661
Date: October 29 & 30 (morning) and November 10 2014 - now with Appendices D and E
Appendix A: Restaurant notes d.d. October 27 2014
Appendix B: Edixhoven's proper theorem and proof d.d. October 28 2014
Appendix C: "A Logic of Exceptions" (ALOE) 2nd edition (2011) part of page 239:
http://thomascool.eu/Papers/ALOE/ALogicOfExceptions.pdf and link to "Occam":
http://thomascool.eu/Papers/ALOE/2012-03-26-CCPO-PCWA.pdf
Alternative approach "Occam versus Cantor": N ~ R.
Appendix D: Proof A in ZFC; Appendix E: Retraction of point (3)
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(1) Edixhoven has provided on October 30 (night) proof that set A satisfies the ZFC axioms. Colignatus only had time to look into this on November 10, and would agree if (2) is clarified.
PM 1. With *f* the identity function, A becomes the Russell set. Thus potentially there is a risk of that paradox. See Appendix E for a effort to find such an example, now retracted however.
PM 2. For X = N the set of natural numbers, while assuming that numbers are not sets, the identity function does not apply here, and we must look for logical errors in Cantor's original diagonal argument.

de verz. deelv. van X

Stelling. Laat X een verzameling zijn, en $f: X \to P(X)$ een afbeelding. Dan is f niet surjectief.

Bewijs. We geven een element van P(X) dat niet in het beeld van f zit.
Dat element is: $A := \{x \in X : x \notin f(x)\}$.
Stel namelijk dat A wèl in het beeld zit, namelijk dat $y \in X$ en $f(y) = A$.

Prove that A is in ZFC (Appendix D). Assume that ZFC always provides for well-defined sets.

Er zijn 2 gevallen: $y \in A$, en $y \notin A$.
Als $y \in A$, dan $y \notin f(y)$, dus $y \notin A$, tegenspraak.
Als $y \notin A$, dan $y \in f(y)$, dus $y \in A$, tegenspraak.
Beide gevallen geven een tegenspraak, dus de aanname dat A in het beeld van f zit is onjuist.
Conclusie: A zit niet in het beeld. $\boxtimes$

Or ZFC does not always provide for well-defined sets.

(2) There is also a B. It will be useful when Edixhoven can explain the relation between aboven A and this B.

It is inefficient to replace A by B everywhere, so I write A for B.

Edixhoven said that the Paul of Venice consistency criterion should not affect the deduction, so he will not mind to take B = A.

But then there is no proof.

If he still holds that there is a proof, then he should explain the relation between this A (A = B) and the A above.

Schrijf B = A

de verz. deelv. van X

||

Stelling. Laat X een verzameling zijn, en $f: X \to P(X)$ een afbeelding. Dan is f niet surjectief.

Bewijs. We geven een element van $P(X)$ dat **misschien** niet in het beeld van f zit.

Dat element is: $A := \{ x \in X : x \notin f(x)$ **& x in A }**

Stel namelijk dat A wèl in het beeld zit, namelijk dat $y \in X$ en $f(y) = A$.

Er zijn 2 gevallen: $y \in A$, en $y \notin A$.

Als $y \in A$, dan $y \notin f(y)$, dus $y \notin A$ **& y in A**, tegenspraak.

Als $y \notin A$, dan $y \in f(y)$ **of y niet in A. Geen tegenspraak.**

Alleen (y in A) geeft een tegenspraak, dus de aanname dat A in het beeld van f zit **geeft (y niet in A), en verder niets**

Conclusie: **Met een welgedefinieerde A is de stelling niet te bewijzen**

This uses a small consistency enhancer invented by Paul of Venice (1368-1428), see I.M. Bochenski "History of Formal Logic".

See ALOE and my text on Logicomix: http://thomascool.eu/Papers/ALOE/2010-02-14-Russell-Logicomix.pdf

It is conceivable that there are other ways to prove the "theorem". However, it is more sensible to look at threevalued logic, since this is already available given the discussion on the Liar Paradox.

It is still possible that Cantor's original proof for N and R survives. However, see the rejection in http://thomascool.eu/Papers/ALOE/2012-03-26-CCPO-PCWA.pdf
The alternative approach of "Occam versus Cantor" is: N ~ R.

Appendix A: Restaurant notes of October 27 2014

27/10/2014 Jan Bergstra, Thomas Cool / & Bas Edixhoven

powerset van X, de verz. van deelv. van X.

<u>Stelling</u>. Laat X een verz. zijn, $f : X \to P(X)$ een afbeelding. Dan is f niet surjectief.

<u>Bewijs</u>. We geven een element van $P(X)$ dat niet in het beeld van f zit.
Dat element is: $A := \{x \in X : x \notin f(x)\}$.
Stel namelijk dat A in het beeld zit, namelijk dat $y \in X$ en $f(y) = A$. Dan $y \notin f(y)$, maar dat betekent dat $y \notin A$, want $f(y) = A$. Maar dan, vanwege de definitie van A: $y \in f(y)$. Dat is in tegenspraak met $y \notin f(y)$.
Conclusie: A zit <u>niet</u> in het beeld. $\boxtimes$

y

B

$B = \{ x \in X : x \notin f(x) \;\&\; x \in X \}$

Paul ve Venice

$x \in B \iff \{ x \in B \;\&\; x \notin f(x) \}$

$\mathbb{R}$

$2^N \to 2^N$

Appendix B: Edixhoven, October 28 2014, proper proof

de verz. deelv.
van X

Stelling. Laat X een verzameling zijn, en $f : X \to \mathcal{P}(X)$ een afbeelding. Dan is f niet surjectief.

Bewijs. We geven een element van $\mathcal{P}(X)$ dat niet in het beeld van f zit.
Dat element is: $A := \{x \in X : x \notin f(x)\}$.
Stel namelijk dat A wèl in het beeld zit, namelijk dat $y \in X$ en $f(y) = A$.
Er zijn 2 gevallen: $y \in A$, en $y \notin A$.
Als $y \in A$, dan $y \notin f(y)$, dus $y \notin A$, tegenspraak.
Als $y \notin A$, dan $y \in f(y)$, dus $y \in A$, tegenspraak.
Beide gevallen geven een tegenspraak, dus de aanname dat A in het beeld van f zit is onjuist.
Conclusie: A zit niet in het beeld. $\boxtimes$

Appendix C. ALOE part of page 239

Paper on Occam March 2012 and update:
http://thomascool.eu/Papers/ALOE/2012-03-26-CCPO-PCWA.pdf

ALOE 2011, p 239, http://thomascool.eu/Papers/ALOE/ALogicOfExceptions.pdf

Wallace (2003:275)) is as follows. Let f: $A \to 2^A$ be the hypothetical bijection between A and its power set. Let $\Phi = \{x \in A \mid x \notin f[x]\}$. Clearly Φ is a subset of A and thus there is a $\varphi = f^{-1}[\Phi]$ so that $f[\varphi] = \Phi$. The question now arises whether $\varphi \in \Phi$ itself. We find that $\varphi \in \Phi \Leftrightarrow \varphi \notin f[\varphi] \Leftrightarrow \varphi \notin \Phi$ which is a contradiction. Ergo, there is no such f. This completes the currently existing proof of Cantor's theorem. The subsequent discussion is to show that this proof cannot be accepted.

11.4.3 Rejection of this proof

Similar like with Russell's set paradox (see above) we might hold that above Φ is badly defined since it is self-contradictory under the hypothesis. A badly defined 'something' may just be a weird expression and need not represent a true set. A test on this line of reasoning is to insert a small consistency condition, giving us $\Phi = \{x \in A \mid x \notin f[x] \wedge x \in \Phi\}$. Now we conclude that $\varphi \notin \Phi$ since it cannot satisfy the condition for membership, i.e. we get $\varphi \in \Phi \Leftrightarrow (\varphi \notin f[\varphi] \wedge \varphi \in \Phi) \Leftrightarrow (\varphi \notin \Phi \wedge \varphi \in \Phi) \Leftrightarrow$ *falsum*. Puristically speaking, the Φ defined in 11.4.2 differs lexically from the Φ defined here, with the first expression being nonsensical and the present one consistent. It will be useful to reserve the term Φ for the proper definition and use Φ' for the expression in 11.4.2. The latter symbol is part of the lexical description but does not meaningfully refer to a set. Using this, we can also use $\Phi^* = \Phi \bigcup \{\varphi\}$ and we can express consistently that $\varphi \in \Phi^*$. So the "proof" above can be seen as using a confused mixture of Φ and Φ^*.

It follows:

1. that the proof for Cantor's Theorem (i.e. as used above) is based upon a badly defined and inherently paradoxical construct, and that this proof evaporates once a sound construct is used.

Appendix D: Proof for (1) that A is in ZFC

Edixhoven, Thu, 30 Oct 2014 22:32:08 +0100:

> Ik gebruik de formulering zoals op
> http://en.wikipedia.org/wiki/Zermelo%E2%80%93Fraenkel_set_theory
>
> De definitie is: A = {x in X : x not in f(x)}.
>
> Om dit in verzamelingentaal te schrijven: merk op dat f een deelverzameling is van X x P(X) is.
> P(X) bestaat vanwege axioma 8.
> X x P(X) bestaat vanwege axioma 4.
>
> A bestaat vanwege axioma 3, namelijk:
>
> A = {x in X : (exists y in P(X)): ((x,y) in f) and(x not in y)}.

Colignatus, November 10:

Thank you for this reminder. Apparently I had no longer in active memory that ZFC blocks the Russell Paradox by the axiom of separation, and that this is an instance of this.

I would agree that the proof then holds in ZFC.

However, my point in the restaurant was different: there is issue (2), how does this A relate to said B ?

You hold that you restrict yourself to ZFC, and from that point you would argue that "a proof in ZFC is a proof in ZFC". However, I hope that you agree that question (2) still is valid, and creates a serious issue for ZFC.

The proper conclusion could well be that ZFC still uses a handicapped definition of "well formed set", so that the "proof" still generates noise w.r.t. better notions of "well formed sets".

> In particular, using the same wikipedia for their discussion of Cantor's theorem for N: this has a set D of "non-selfish numbers", and there the "proof" evaporates with the consistency condition by Paul of Venice:
>
> http://en.wikipedia.org/wiki/Cantor%27s_theorem
>
> D* = {x in N | x is a non-selfish number & x in D* }
>
> (x in D*) iff ((x in N) & (x is non-selfish) & (x in D*))
>
> Let d be the number for D*: (d is non-selfish) translates as (d not in D*)
>
> (d in D*) iff ((d in N) & (d not in D*) & (d in D*))
>
> Ergo: (d not in D*)
>
> No paradox, no proof for N < P(N)

Appendix E: Retracted example of f = identity function

(3) W.r.t. point (1) it might be useful to develop a case where f can be the identity function so that Russell's set paradox comes into view. Potentially this highlights the paradoxical structure of the argument. I presented this example on October 30 with the idea that it would clarify the issue, but now on November 10 retract it. For the record I keep it here, now with an explanation why I retract it. The reason is that ZFC already provides a protection against the Russell paradox, so that its paradoxical effect shows from (2) and not from here.

Consider sets Z and Y, such that there is a X C Z so that P[X] C Y, or, that Y contains at least a powerset of a subset of Z. If X = Z then P[Z] would be a subset for Y.

An example to think of is Z = Y = N U P[N] or the natural numbers N and their power set.

Adjusted Theorem: Consider sets Z and Y, such that there is a X C Z so that P[X] C Y.
Let f be a function f: Z -> Y, then f is not surjective.

Counter-example for this Adjusted Theorem: Take f = identity function and Y = Z = N U P[N]. Then obviously f is surjective. Every element in Y has an element in Z such that y = I[z]. But the Edixhoven proof would cause a different conclusion. Thus that proof cannot be used for this Adjusted Theorem. Only a part can be used, namely (y not in X) so that (y in Z \ X).

The given A belongs to Y, since A would be in P[X].
The proper proof uses (y in Z), with two subcases: (y in X) or (y in Z \ X).
The Edixhoven steps only give the decision that (y not in X).
They cause the conclusion that (y in Z \ X), and not yet that there is no surjection.
In the counter-example f = identity and Y = Z = N U P[N] we find: (y in P[X]).
Since no element of X = N would contain itself, y = A = N.

We copy the Edixhoven proof - quite aware that it was created with another purpose.
(The original proof uses additional assumptions X = Z and Y = P[Z], that we deviate from.)

Bewijs. We geven een element van **Y** dat niet in het beeld van f zit.
Dat element is: $A := \{x \in X : x \notin f(x)\}$.
Stel namelijk dat A wèl in het beeld zit, namelijk dat $y \in X$ en $f(y) = A$.
Er zijn 2 gevallen: $y \in A$, en $y \notin A$.
Als $y \in A$, dan $y \notin f(y)$, dus $y \notin A$, tegenspraak.
Als $y \notin A$, dan $y \in f(y)$, dus $y \in A$, tegenspraak.
Beide gevallen geven een tegenspraak, dus de aanname dat A in het beeld van f zit is onjuist.
Conclusie: A zit niet in het beeld. ⊠

In the counter-example with f = identity we find y = f(y) = A = {x in X: x not in x} which is the Russell set (restricted to X). ZFC does not prohibit this use since the axiom of separation has the (x in X) qualifier.

We find: (y in y) iff ((y in X) & (y not in y)), so that (y not in X). This proof only generates that (y in Z \ X). In the counter-example (y in P[X]).

The idea behind creating this counter-example was that there is an obvious surjective case, while the "proof" suggests there would not be one. However, the counter-example holds with respect to the Adjusted Theorem and not to Cantor's Theorem.

The counter-example starts biting into Cantor's Theorem - but this would have to be looked into:

(a) If we consider tiny Z \ X and Y \ P[X] so that there is only a margial difference.

(b) If there would be a deduction from (y in Z \ X) into some inconsistency.

PM 1. ZFC does not allow (A in A) but it is a bit vague whether one is allowed to use such a set in counterfactual fashion. Thus perhaps ((A in A) => falsum) might perhaps be allowed, since it causes the conclusion that NOT (A in A).

PM 2. An erroneous argument of mine was: "Thus it is important to insert the statement "And assume that A is in ZFC", so that the proper conclusion is that A isn't in ZFC." This is now resolved by Edixhoven showing that A would be in ZFC. The line of reasoning would remain important however for other axioms on "well defined sets".